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ON THE THEORY OF RADIATION FROM SOURCES IN  
PLANE STRATIFIED ANISOTROPIC PLASMA MEDIA

by

E. Arbel and L. B. Felsen

Research Report No. PIBMRI-1069-62

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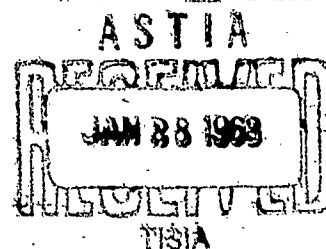
United States Air Force

L. G. Hanscom Field

Bedford, Massachusetts

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ON THE THEORY OF RADIATION FROM SOURCES  
IN PLANE STRATIFIED ANISOTROPIC PLASMA MEDIA

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While it was originally intended that the present authors prepare a condensed and revised version of part of the thesis, this project was delayed by the necessity of Dr. Arbel's return to his permanent residence in Israel. To make the timely results of the research available, it was decided to issue in an interim report the unchanged thesis entitled "Radiation from a Point Source in an Anisotropic Medium" by E. Arbel, Report PIBMRI-861-60, Microwave Research Institute, Polytechnic Institute of Brooklyn, November 1960. The present report is the first of two which contain the previously planned condensation and revision of this work.

Abstract

This report deals with the radiation from arbitrary source distributions in plane stratified, anisotropic media. The formal solutions are obtained by an extension of modal procedures familiar from the analysis of isotropic waveguide regions. Special attention is given to the formulation of a radiation condition which requires the flow of energy away from the sources, and to its interpretation utilizing the refractive index surfaces descriptive of plane wave propagation in an anisotropic medium. These concepts are illustrated in detail for an ionized plasma under the influence of an external steady magnetic field.

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## I. Introduction

The propagation of plane electromagnetic waves in regions with anisotropic permittivity or permeability has received considerable attention in the literature – the former in connection with the study of the optical properties of crystals<sup>1</sup> and the electromagnetic behavior of the ionosphere<sup>2, 3</sup>, and the latter in connection with magnetized ferrites<sup>4</sup>. In these applications, the sources of the field have been of minor concern, and emphasis has been placed on the determination of the wave spectrum in the region. However, some recent problems – for example, the investigation of discontinuities in ferrite or plasma loaded waveguides, Cerenkov effects due to moving charged particles in a magneto-ionic medium<sup>5</sup>, or radio communication from satellites passing through the ionosphere — require a knowledge of the radiation characteristics of localized sources.

This report deals with the radiation from arbitrary time harmonic source distributions (varying like  $\exp(j\omega t)$ ) in a transversely unbounded region filled with a dielectric medium characterized by a tensor permittivity  $\underline{\epsilon}$  as shown in (1). The permeability  $\mu_0$  is assumed to be constant.  $\underline{\epsilon}$  may be a piecewise constant function of the longitudinal variable  $z$ , thereby admitting linear stratification. By an extension of the modal procedure described for isotropic regions by Marcuvitz and Schwinger<sup>6</sup>, the transverse (to  $z$ ) electromagnetic fields  $\underline{E}_t(x, y, z)$  and  $\underline{H}_t(x, y, z)$  are represented as a superposition of orthogonal transverse vector mode fields having the form  $V_i(z) \underline{e}_i(x, y)$  and  $I_i(z) \underline{h}_i(x, y)$ , respectively; each mode is a solution of the source-free Maxwell field equations. Because of the infinite extent of the cross-sectional ( $x - y$ ) domain and the non-variability of  $\underline{\epsilon}$  with  $x$  and  $y$ , the transverse vector mode functions  $\underline{e}_i$  and  $\underline{h}_i$  have an exponential dependence of the form  $\exp(j\xi x + j\eta y)$ , where the modal index  $i \rightarrow (\xi, \eta)$  is continuously variable. As regards their vectorial characteristics, the modes are found to separate into two sets, "ordinary"

and "extraordinary" (see Appendix A), in terms of which the electromagnetic fields can be represented as in (7a, b). The modal amplitudes  $V$  and  $I$  are found to satisfy the first-order differential equations (8) (transmission line equations), thereby permitting the use of network methods in their determination. The continuity requirements of the tangential electric and magnetic fields across an interface between two different anisotropic regions, or the imposition of boundary conditions at a terminal surface of the region, give rise to coupling between the ordinary and extraordinary modal amplitudes. Their systematic calculation via equivalent network procedures is emphasized in Sec. II.

The transmission line equations (8) descriptive of the spatial behavior of  $V$  and  $I$  in each layer contain as parameters the propagation constant  $\mathcal{K}$  and the characteristic impedance  $Z$ . As shown in Appendix A,  $\mathcal{K}$  in an anisotropic medium is generally a complicated multivalued function of the transverse wave-numbers ( $\xi$ ,  $\eta$ ). Its proper definition on the integration path, essential for a unique specification of the fields via the modal representations (7a, b), is accomplished by recourse to a radiation condition which requires that the energy flow due to a localized source distribution is outward from the source region.<sup>7</sup> The associated restrictions on  $\mathcal{K}$ , complicated by the fact that the directions of energy and phase propagation generally differ in an anisotropic medium, are discussed in Sec. III -- both analytically, and through use of the refractive index surfaces for the medium which specify the variation in refractive index as a function of the direction of propagation of a plane wave. Special attention is given to the case of a plasma under the influence of a longitudinal d. c. magnetic field, and to the effect of the plasma parameters in determining the propagation characteristics of electromagnetic waves. The definition of the multivalued function  $\mathcal{K}$  for various plasma parameters is examined in detail in Sec. IV, thereby rendering unique the formal solution of this class of radiation problems.

The results of this analysis have been applied elsewhere<sup>8, 9</sup> to the detailed study of the radiation field of an electric current element in an infinite, and semi-infinite, homogeneous, anisotropic plasma medium.

Investigations by other authors have been concerned primarily with the study of radiation from infinite, semi-infinite, and single-slab anisotropic plasma regions. For orientations of the gyrotropic axis perpendicular to the interface, the formal solutions for these configurations emerge as special cases of our analysis. While the imposition of a radiation condition is essential in the formulation of radiation problems in infinite and semi-infinite regions, the detailed discussion of its effect on the proper specification of the representation

integrals as carried out herein has generally been omitted by other authors. With reference to the infinite medium problem we cite the work of Abraham<sup>10</sup>, Bunkin<sup>11</sup>, Kogelnik<sup>12</sup>, Kuehl<sup>13</sup>, Mittra<sup>14</sup>, Mittra and Deschamps<sup>15</sup>, Clemmow<sup>16</sup>, Motz and Kogelnik<sup>17</sup>, Chow<sup>18</sup>. Abraham\* has employed a two-dimensional Fourier integral representation for the fields similar to that presented here, and has also pointed out the utility of the refractive index surfaces in dealing with the radiation condition; his analysis in connection with the latter does not, however, enter into the function-theoretic questions treated in Secs. III and IV. The approach of the other authors (except Clemmow) differs in that they proceed via a three-dimensional Fourier integral formulation which, though imbued with a certain formal elegance for the infinite medium problem, appears less convenient for an asymptotic analysis of the radiation field, and also seems not directly suited to the study of stratified media. Radiation from dipole sources in the presence of an anisotropic plasma half-space has been analyzed by Barsukov<sup>19</sup>, Arbel<sup>8</sup>, and Tyras, Ishimaru, and Swarm<sup>20</sup>. Hodara<sup>21</sup> has recently been concerned with the problem of radiation through an anisotropic plasma slab; he employs a two-dimensional Fourier transform procedure but omits any function-theoretic discussion of the type mentioned above. It also seems that the matching of the boundary conditions at the slab interface and at the source could be more easily achieved by the network procedure described here, as may be seen from the analysis of Wu<sup>22</sup> who has used the network approach for the slab problem. Several studies of the simpler problems of radiation by specially oriented line sources in a gyrotropic medium<sup>23, 24, 25</sup>, or of radiation in a plasma subjected to weak<sup>20</sup> ( $\omega_c \rightarrow 0$  in (35)) or strong<sup>13, 26, 27, 28, 29</sup> ( $\omega_c \rightarrow \infty$ ) external magnetic field have likewise been carried out.

Green's function representations for general anisotropic waveguides have been obtained by Bresler and Marcuvitz<sup>30</sup> via abstract operator methods. For the special class of problems herein, our procedure, though less general, yields the result more directly and by conventional methods.

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\* The authors are indebted to G. Meltz of the Air Force Cambridge Research Laboratories for calling this reference to their attention during the preparation of this manuscript.

## II. Formal Solution

### a. Reduction of the field equations

Consider an arbitrary (but prescribed) distribution of time-harmonic sources of electric current  $\underline{J}(\underline{r})$  and magnetic current  $\underline{M}(\underline{r})$  in a medium comprising a series of transversely unbounded, parallel anisotropic layers (Fig. 1). A coordinate system is chosen so that the z-axis is perpendicular to the layer

( See Fig. 1 in Appendix C )

interfaces, and each layer is assumed to be characterized by a scalar permeability  $\mu_0$  and a tensor dielectric constant

$$\hat{\underline{\epsilon}}^{(a)} = \hat{\underline{\epsilon}}_t^{(a)} + \underline{z}_0 \underline{z}_0 \hat{\epsilon}_z^{(a)}, \quad \hat{\underline{\epsilon}}_t^{(a)} \rightarrow \begin{pmatrix} \hat{\epsilon}_1^{(a)} & j\hat{\epsilon}_2^{(a)} \\ -j\hat{\epsilon}_2^{(a)} & \hat{\epsilon}_1^{(a)} \end{pmatrix}, \quad (1)$$

where  $^{(a)}$  denotes the  $a$ -th layer,  $\hat{\underline{\epsilon}}_t$  is a transverse (to  $z$ ) dyadic whose representative in an  $x$ - $y$  coordinate system is as shown, and  $\underline{z}_0$  is a unit vector along the  $z$ -direction. These dielectric tensors have the form appropriate to an ionized plasma medium under the influence of a d.c. magnetic field along the  $z$ -axis. We seek a solution of the steady-state Maxwell field equations

$$\nabla \times \underline{E}^{(a)} = -j\omega\mu_0 \underline{H}^{(a)} - \underline{M}^{(a)}, \quad \nabla \times \underline{H}^{(a)} = j\omega\hat{\underline{\epsilon}}^{(a)} \cdot \underline{E}^{(a)} + \underline{J}^{(a)}, \quad (2)$$

in each layer, subject to the required continuity of the transverse electromagnetic fields  $\underline{E}_t^{(a)}$  and  $\underline{H}_t^{(a)}$  at each interface, to prescribed boundary conditions at the  $z$ -termini of the region (if any), and to a radiation condition at infinity. A time variation  $\exp(j\omega t)$  is understood throughout.

If the configuration in Fig. 1 is viewed as a waveguide with axis along  $z$ , the formal solution of the boundary value problem in (2) can be effected by an application of guided wave techniques developed for isotropic and anisotropic regions.<sup>6, 30, 31</sup> First, by taking the scalar and vector products of Eqs. (2) and the longitudinal unit vector  $\underline{z}_0$ , one derives after some rearrangement the equations for the transverse field components (note:  $\underline{z}_0 \times \underline{1} \cdot \hat{\underline{\epsilon}}_t = \hat{\underline{\epsilon}}_t \cdot \underline{z}_0 \times \underline{1}$ ),

$$-\frac{\partial \underline{E}_t}{\partial z} = j\omega\mu_0 \left[ \underline{1}_t + \frac{\nabla_t \nabla_t}{k^2} \right] \cdot \underline{H}_t \times \underline{z}_0 + \hat{\underline{M}}_t \times \underline{z}_0, \quad (3a)$$

$$-\frac{\partial \underline{H}_t}{\partial z} = j\omega\epsilon \left[ \underline{\epsilon}_t + \frac{\nabla_t \nabla_t}{k^2} \right] \cdot \underline{z}_0 \times \underline{E}_t + \underline{z}_0 \times \hat{\underline{J}}_t, \quad (3b)$$

from which the longitudinal components can be obtained via the relation,

$$\underline{E}_z = \frac{1}{j\omega\epsilon} \left[ \nabla_t \cdot \left( \underline{H}_t \times \underline{z}_0 \right) - J_z \right], \quad \underline{H}_z = \frac{1}{j\omega\mu_0} \left[ \nabla_t \cdot \left( \underline{z}_0 \times \underline{E}_t \right) - \underline{M}_z \right]. \quad (3c)$$

To simplify the notation, the superscripts<sup>(a)</sup> identifying the various layers have been omitted, and the following definitions have been introduced:

$$\underline{\epsilon}_t = \frac{1}{\epsilon} \hat{\underline{\epsilon}}_t \rightarrow \begin{pmatrix} \epsilon_1 & j\epsilon_2 \\ -j\epsilon_2 & \epsilon_1 \end{pmatrix}, \quad \hat{\underline{\epsilon}}_z \equiv \epsilon = \epsilon_0 \epsilon_z, \quad k^2 = k_0^2 \epsilon_z, \quad k_0^2 = \omega^2 \mu_0 \epsilon_0, \quad (3d)$$

where  $\epsilon_0$  is the dielectric constant for free space,  $\epsilon_z$  is the normalized longitudinal dielectric constant  $\hat{\epsilon}_z/\epsilon_0$ , and  $\epsilon_{1,2} = \epsilon_{1,2}/\epsilon$  are normalized to  $\epsilon$ . \*  $\nabla_t = (\nabla - \underline{z}_0 \partial/\partial z)$  represents the transverse gradient operator,  $\underline{1}_t$  the transverse unit dyadic, and

$$\hat{\underline{J}}_t = \underline{J}_t - \frac{1}{j\omega\mu} \underline{z}_0 \times \nabla_t \underline{M}_z, \quad \hat{\underline{M}}_t = \underline{M}_t + \frac{1}{j\omega\epsilon} \underline{z}_0 \times \nabla_t J_z, \quad (3e)$$

are equivalent transverse electric and magnetic source current distributions.

Next, we seek a representation of the transverse field components in terms of a set of transverse vector eigenfunctions which are complete, orthogonal, and individually satisfy the homogeneous field equations and the required boundary conditions, in the transverse domain. As shown in Appendix A, the electric mode

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\* For the plasma case, normalization to  $\epsilon$  implies that  $\underline{\epsilon}_t = \underline{1}_t$  in the absence of an external d. c. magnetic field (see Sec. IV).

functions  $\underline{e}$  and the magnetic mode functions  $\underline{h}$  comprise a continuous distribution of plane waves whose vectorial characteristics can be grouped into two categories, "ordinary" and "extraordinary", to be denoted by the subscripts o and e, respectively:

$$\underline{e}_{o,e}(x,y;\xi,\eta) = \underline{e}_{o,e}(\xi,\eta) \frac{k e^{-jk(\xi x + \eta y)}}{2\pi}, \quad -\infty < (\xi,\eta) < \infty, \quad (4a)$$

$$\underline{h}_{o,e}(x,y;\xi,\eta) = \underline{h}_{o,e}(\xi,\eta) \frac{k e^{-jk(\xi x + \eta y)}}{2\pi}, \quad (4b)$$

where

$$\underline{e}_e = \frac{\sigma^2 + \Delta}{\Delta \sqrt{2} \sigma} \underline{\sigma} + \frac{j\delta \sqrt{2}}{\Delta \sigma} (\underline{\sigma} \times \underline{z}_o), \quad \underline{\sigma} = \underline{x}_o \xi + \underline{y}_o \eta, \quad (4c)$$

$$\underline{h}_e \times \underline{z}_o = \frac{1}{\sqrt{2} \sigma} \underline{\sigma} + j \frac{\sigma^2 + \Delta}{2 \sqrt{2} \delta \sigma} (\underline{\sigma} \times \underline{z}_o), \quad \sigma^2 = \xi^2 + \eta^2, \quad (4d)$$

and

$$\delta = \frac{\epsilon_2}{\epsilon_1 - 1}, \quad \Delta = \sqrt{\sigma^4 + 4\delta^2(1 - \sigma^2)} \operatorname{sgn}(\epsilon_1 - 1). \quad (4e)$$

While the eigenvalue problem could have been treated for general dissipative media, <sup>30,22</sup> the above analysis has been restricted to the lossless case for which  $\epsilon$ ,  $\epsilon_1$  and  $\epsilon_2$  are real. It is noted that the wavenumbers  $\xi$  and  $\eta$  have been normalized to  $k$ , thereby implying that  $\epsilon > 0$  and  $k > 0$ . The modifications introduced when  $\epsilon < 0$  are discussed at the end of Appendix A. The orthogonality properties of the position vectors  $\underline{e}_{o,e}$  and  $\underline{h}_{o,e}$  differ according to whether  $\Delta$  is real or imaginary (the square root in (4e) is defined to be positive when real and negative imaginary otherwise):

$$\begin{aligned} \underline{e}_o^* \cdot \underline{h}_o \times \underline{z}_o &= 1, & \underline{e}_e^* \cdot \underline{h}_e \times \underline{z}_o &= 0, & \Delta &\text{real}, \\ \underline{e}_o^* \cdot \underline{h}_e \times \underline{z}_o &= 0, & \underline{e}_e^* \cdot \underline{h}_o \times \underline{z}_o &= 1, & \Delta &\text{imaginary}, \end{aligned} \quad (5)$$

where the asterisk denotes the complex conjugate. Thus, the mode functions in (4a, b) satisfy the bi-orthogonality relation,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \underline{e}^*(x, y; \xi, \eta) \cdot \underline{h}(x, y; \xi', \eta') \times \underline{z}_0 = \underline{e}^* \cdot \underline{h} \times \underline{z}_0 \delta(\xi - \xi') \delta(\eta - \eta') , \quad (6)$$

which, upon insertion of the appropriate modal subscripts o or e, is simplified further from (5).

The various vector functions in (3a, b) can now be represented as follows:

$$\underline{E}_t(x, y, z) = \frac{k}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left[ V_o(z; \xi, \eta) \underline{e}_o(\xi, \eta) + V_e(z; \xi, \eta) \underline{e}_e(\xi, \eta) \right] e^{-jk(\xi x + \eta y)}, \quad (7a)$$

$$\underline{H}_t(x, y, z) = \frac{k}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left[ I_o(z; \xi, \eta) \underline{h}_o(\xi, \eta) + I_e(z; \xi, \eta) \underline{h}_e(\xi, \eta) \right] e^{-jk(\xi x + \eta y)}, \quad (7b)$$

$$\underline{z}_0 \times \hat{\underline{J}}_t(x, y, z) = \frac{k}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left[ i_e(z; \xi, \eta) \underline{h}_e(\xi, \eta) + i_o(z; \xi, \eta) \underline{h}_o(\xi, \eta) \right] e^{-jk(\xi x + \eta y)}, \quad (7c)$$

$$\hat{\underline{M}}_t(x, y, z) \times \underline{z}_0 = \frac{k}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \left[ v_e(z; \xi, \eta) \underline{e}_e(\xi, \eta) + v_o(z; \xi, \eta) \underline{e}_o(\xi, \eta) \right] e^{-jk(\xi x + \eta y)}. \quad (7d)$$

The completeness and existence of these representations (for spatially confined source distributions) in the (x-y) function space follows from the theory of the Fourier integral and from the field behavior at infinity, while that in the 2x2 vector space is assured by the linear combination of the e- and o- eigenvectors. Upon substituting Eqs. (7a-d) into (3a, b), interchanging the orders of differentiation and integration and noting that the operator  $\nabla_t$  can then be replaced by  $-jk\underline{\sigma}$ , one may equate the resulting Fourier transforms on both sides of these equations. Dot product multiplication of the first and second of these equations with  $(\underline{h}_{o,e}^* \times \underline{z}_0)$  and  $(\underline{z}_0 \times \underline{e}_{o,e}^*)$ , respectively, and use of Eqs. (A4) in Appendix A and of the orthogonality relations (5) then yields the following expressions to be satisfied by the transforms  $V_{o,e}$  and  $I_{o,e}$ :

$$- \frac{dV}{dz} = jk \mathcal{Z} I + v, \quad - \frac{dI}{dz} = jk \mathcal{Y} V + i. \quad (8)$$

These equations, valid separately for both the o- and e- modes (for real or imaginary  $\Delta$ ), can evidently be interpreted as transmission line equations, with V, I,  $\mathcal{Z}$ , and Y playing the role of voltage, current, normalized propagation constant, characteristic impedance and characteristic admittance,

respectively, while  $v$  and  $i$  represent voltage and current generator distributions (Fig. 1(b)).  $\mathcal{X}$  and  $Z = 1/Y$  are determined from the eigenvalue problem in Appendix A as

$$\mathcal{X}_e = \sqrt{U \pm \sqrt{U^2 - W}}, \quad Y_e = \mathcal{X}_e \frac{4\delta^2}{\xi(\Delta^2 \pm \sigma^2 \Delta)}, \quad \xi = \sqrt{\frac{\mu_0}{\epsilon}}, \quad (8a)$$

where

$$U = \epsilon_1 - \frac{\epsilon_1 + 1}{2} \sigma^2, \quad W = \epsilon_1(1 - \sigma^2)(\sigma_2^2 - \sigma^2), \quad \sigma_2^2 = \frac{\epsilon_1^2 - \epsilon_2^2}{\epsilon_1}. \quad (8b)$$

$\sqrt{U^2 - W}$  can alternatively be written as  $[\epsilon_1 - 1](\Delta/2)$ . The voltage and current generator strengths are evaluated by inverting Eqs. (7c, d) via (6):

$$i_e = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \underline{e}_e^* \cdot \hat{\underline{J}}_t, \quad v_e = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \underline{h}_e^* \cdot \hat{\underline{M}}_t, \quad \Delta \text{ real}; \quad (9)$$

For imaginary  $\Delta$ ,  $\underline{e}_{e,e}^*$  and  $\underline{h}_{e,e}^*$  are replaced by  $\underline{e}_{e,o}^*$  and  $\underline{h}_{e,o}^*$ , respectively.\*\*

From the two-dimensional divergence theorem

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \underline{A} \cdot \nabla_t f = - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy f \nabla_t \cdot \underline{A} + \oint_s ds f \underline{A} \cdot \underline{n},$$

where  $\underline{A}$  and  $f$  are suitably continuous vector and scalar functions,  $s$  denotes a contour bounding the transverse cross-section at infinity in the  $x$ - $y$  plane and  $\underline{n}$  is a unit vector normal to  $s$ , one observes that if  $f = 0$  on  $s$ , the double integral on the left-hand side is equal to the double integral on the right-hand side. Upon applying this result to Eq. (9) (after substituting (3e) and assuming that  $\underline{J}_z$  and  $\underline{M}_z$  are spatially confined sources), one obtains alternatively,

$$i_e = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{J}_t \cdot \underline{e}_e^* dx dy - \frac{k}{\omega \mu} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{M}_z \underline{e}_e^* \cdot \underline{\sigma} \times \underline{z}_o dx dy, \quad (9a)$$

$$v_e = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{M}_t \cdot \underline{h}_e^* dx dy - \frac{k}{\omega \epsilon} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{J}_z \underline{h}_e^* \times \underline{z}_o \cdot \underline{\sigma} dx dy, \quad (9b)$$

which result can now be applied also to discontinuous current distributions.

The proper definition of the multivalued function  $\mathcal{X}(\eta)$ , and the disposition of the integration path in (7a, b) in relation to its singularities, will

\*\* One observes from Eqs. (4c, d) that the analytic continuation of the functions  $\underline{e}_{e,e}^*$  and  $\underline{h}_{e,e}^*$  from the domain of real  $\Delta$  into the domain of imaginary  $\Delta$  yields the functions  $\underline{e}_{e,o}^*$  and  $\underline{h}_{e,o}^*$ , respectively, evaluated for imaginary  $\Delta$ . Thus, it suffices to evaluate the discontinuously represented inner products in (9) for real  $\Delta$  and continue these functions into the imaginary- $\Delta$  domain. The same is true for other inner products encountered later on.



be discussed further in Sec. III. For the present we note only that  $\mathcal{K}_o$  and  $\mathcal{K}_e$  are associated, respectively, with the + and - signs in (8a), and that  $\sqrt{U^2 - W}$  is defined to be positive when real. When  $\mathcal{K}$  has an imaginary part, the associated wave is non-propagating (along z), while  $\mathcal{K}$  real represents a propagating wave. Since i and v are specified in terms of known quantities, the solution of the Maxwell field equations (2) has been reduced to the solution of the transmission line equations (8) in each layer, subject to the required continuity conditions at each interface and to the specified boundary conditions at the z-termini of the region.

#### b. Solution of the modal network problem

At an interface between two different anisotropic regions, the required continuity of  $\underline{E}_t$  and  $\underline{H}_t$  in (7a, b) can be achieved by assuring the continuity of their Fourier transforms. If parameters pertaining to the two regions are distinguished by superscripts  $\alpha$  and  $\beta$ , respectively, the boundary conditions are satisfied if

$$k^{(\beta)} \begin{bmatrix} V_o^{(\alpha)} \underline{e}_o^{(\alpha)} + V_e^{(\alpha)} \underline{e}_e^{(\alpha)} \end{bmatrix} = k^{(\alpha)} \begin{bmatrix} V_o^{(\beta)} \underline{e}_o^{(\beta)} + V_e^{(\beta)} \underline{e}_e^{(\beta)} \end{bmatrix}, \quad (10a)$$

$$k^{(\beta)} \begin{bmatrix} I_o^{(\alpha)} \underline{h}_o^{(\alpha)} + I_e^{(\alpha)} \underline{h}_e^{(\alpha)} \end{bmatrix} = k^{(\alpha)} \begin{bmatrix} I_o^{(\beta)} \underline{h}_o^{(\beta)} + I_e^{(\beta)} \underline{h}_e^{(\beta)} \end{bmatrix}, \quad (10b)$$

provided that the normalized transverse wave numbers in the two regions are related via

$$k^{(\alpha)} \xi^{(\alpha)} = k^{(\beta)} \xi^{(\beta)}, \quad k^{(\alpha)} \eta^{(\beta)} = k^{(\beta)} \eta^{(\beta)} \quad * \quad (10c)$$

In view of the orthogonality relation (5), Eqs. (10a, b) can be reduced to the matrix form:

$$\begin{bmatrix} \underline{V}^{(\alpha)} \\ \underline{I}^{(\alpha)} \end{bmatrix} = T_z \begin{bmatrix} \underline{V}^{(\beta)} \\ \underline{I}^{(\beta)} \end{bmatrix}, \quad T_z \rightarrow \begin{bmatrix} t_{\alpha\alpha} & t_{\alpha\beta} \\ t_{\beta\alpha} & t_{\beta\beta} \end{bmatrix}, \quad (11)$$

where  $\underline{V}$  and  $\underline{I}$  are the column vectors

$$\underline{V}^{(i)} \rightarrow \begin{bmatrix} V_o^{(i)} \\ V_e^{(i)} \end{bmatrix}, \quad \underline{I}^{(i)} \rightarrow \begin{bmatrix} I_o^{(i)} \\ I_e^{(i)} \end{bmatrix}, \quad i = \alpha, \beta, \quad (11a)$$

while  $T_z$  is the impedance transfer matrix. The matrix elements  $t_{ij}$ ,  $i, j = \alpha, \beta$ , are themselves 2x2 matrices whose elements are given by:

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\* It is implied that  $\xi$ ,  $\eta$  in the integral representations (7a, b) are replaced throughout by the appropriate  $\xi^{(i)}$ ,  $\eta^{(i)}$ ,  $i = \alpha, \beta$ , whence  $V_{o,e}^{(\alpha)} = V_{o,e}(z; \xi^{(\alpha)}, \eta^{(\alpha)})$ ,  $V_{o,e}^{(\beta)} = V_{o,e}(z; \xi^{(\beta)}, \eta^{(\beta)})$ , etc. This is to be borne in mind when interpreting formulas involving parameters with different superscripts.

$$t_{a\beta} = t_{\beta a} = 0 ,$$

(11b)

$$t_{aa} \rightarrow \frac{k^{(a)}}{k^{(\beta)}} \begin{bmatrix} \underline{h}_o^{(a)*} \times \underline{z}_o \cdot \underline{e}_o^{(\beta)} & \underline{h}_o^{(a)*} \times \underline{z}_o \cdot \underline{e}_e^{(\beta)} \\ \underline{h}_e^{(a)*} \times \underline{z}_o \cdot \underline{e}_o^{(\beta)} & \underline{h}_e^{(a)*} \times \underline{z}_o \cdot \underline{e}_e^{(\beta)} \end{bmatrix} , \quad (11c)$$

$$t_{\beta\beta} \rightarrow \frac{k^{(a)}}{k^{(\beta)}} \begin{bmatrix} \underline{e}_o^{(a)*} \cdot \underline{h}_o^{(\beta)} \times \underline{z}_o & \underline{e}_o^{(a)*} \cdot \underline{h}_e^{(\beta)} \times \underline{z}_o \\ \underline{e}_e^{(a)*} \cdot \underline{h}_o^{(\beta)} \times \underline{z}_o & \underline{e}_e^{(a)*} \cdot \underline{h}_e^{(\beta)} \times \underline{z}_o \end{bmatrix} . \quad (11d)$$

Expressions (11c, d) apply for real or imaginary  $\Delta$  (see footnote on p. 8). Since the submatrices  $t_{aa}$  and  $t_{\beta\beta}$  are not diagonal, the ordinary and extraordinary modes are coupled at the interface. Because  $t_{a\beta} = t_{\beta a} = 0$ , the coupling occurs in a particularly simple manner and can be schematized in terms of the transformer network shown in Fig. 2. It is also noted, in view of the symmetrical character of Eqs. (10) as regards  $a$  and  $\beta$ , that the elements of the inverse matrix  $T_z^{-1}$  are given as in (11c, d) provided that the superscripts  $a$  and  $\beta$  are interchanged throughout.

In the interior of each slab region, the equivalent modal network comprises the two transmission lines representative of the o- and e-modes, respectively, as shown in Fig. 1(b). If the slab in question has a length  $d$ , the impedance transfer matrix  $\hat{T}_z$  for the slab region is obtained from simple transmission line theory as follows:

$$\hat{T}_z \rightarrow \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} , \quad (12)$$

where the 2x2 submatrices  $t_{ij}$ ,  $i, j = 1, 2$ , are diagonal and are given by

$$t_{11} = t_{22} = \cos(k \hat{\mathcal{X}} d) , \quad \hat{Y} t_{12} = \hat{Z} t_{21} = j \sin(k \hat{\mathcal{X}} d). \quad (12a)$$

$\hat{\mathcal{X}}$  and  $\hat{Z}$  are the diagonal propagation constant and characteristic impedance matrices, respectively,

$$\hat{\mathcal{X}} \rightarrow \begin{bmatrix} \mathcal{X}_o & 0 \\ 0 & \mathcal{X}_e \end{bmatrix} , \quad \hat{Z} = \hat{Y}^{-1} \rightarrow \begin{bmatrix} Z_o & 0 \\ 0 & Z_e \end{bmatrix} . \quad (12b)$$

The matrices in (12a) are interpreted by recalling that if  $\hat{\mathcal{Y}}$  is a diagonal matrix, with elements  $\mathcal{Y}_i$ , a matrix  $f(\hat{\mathcal{Y}})$  is also diagonal and has as its elements  $f(\mathcal{Y}_i) \delta_{ij}$ . The voltage and current vectors  $\bar{V}^{(1)}$  and  $\bar{I}^{(1)}$  at the slab face  $z = z_1$  are then related to the analogous quantities at  $z = z_2$  via

$$\begin{bmatrix} \bar{V}^{(1)} \\ \bar{I}^{(1)} \end{bmatrix} = \hat{T}_z \begin{bmatrix} \bar{V}^{(2)} \\ \bar{I}^{(2)} \end{bmatrix}, \quad z_2 - z_1 = d > 0 \quad * \quad (12c)$$

Repeated application of Eqs. (11) and (12) allows one to express the voltages and currents at any point  $z_\alpha$  in the region in terms of the voltages and currents at any other point  $z_\beta$ . The overall transfer matrix descriptive of the network between  $z_\alpha$  and  $z_\beta$  is then composed of the ordered product of the transfer matrices of the network constituents in this region.

If the region is terminated at  $z = z_0$  in a plane surface on which  $\underline{E}_t$  and  $\underline{H}_t$  are related by the boundary condition

$$\underline{E}_t(\underline{\rho}, z_0) = \underline{Z} \cdot \underline{H}_t(\underline{\rho}, z_0) \times \underline{z}_0, \quad (13)$$

where the transverse dyadic  $\underline{Z}$  denotes a constant anisotropic surface impedance whose representative in an x-y coordinate space is

$$\underline{Z} \rightarrow \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}, \quad (13a)$$

then the corresponding relation between  $V_{o,e}$  and  $I_{o,e}$  at  $z = 0$  is given via (7a, b) by

$$V_{o,e} + V_{e,e} = \underline{Z} \cdot (I_{o,h} + I_{e,h}) \times \underline{z}_0. \quad (14)$$

Use of the orthogonality relations (5) allows one to deduce the terminal impedance matrix  $\hat{Z}_s$  for this structure as

$$\begin{bmatrix} V_o \\ V_e \end{bmatrix} = \hat{Z}_s \begin{bmatrix} I_o \\ I_e \end{bmatrix}, \quad (15a)$$

where for real or imaginary  $\Delta$  (see footnote p. 8),

$$\hat{Z}_s \rightarrow \begin{bmatrix} \underline{h}_o^* \times \underline{z}_o \cdot \underline{Z} \cdot \underline{h}_o \times \underline{z}_o & \underline{h}_o^* \times \underline{z}_o \cdot \underline{Z} \cdot \underline{h}_e \times \underline{z}_o \\ \underline{h}_e^* \times \underline{z}_o \cdot \underline{Z} \cdot \underline{h}_o \times \underline{z}_o & \underline{h}_e^* \times \underline{z}_o \cdot \underline{Z} \cdot \underline{h}_e \times \underline{z}_o \end{bmatrix}. \quad (15b)$$

\* In this equation, the superscripts refer to quantities at different  $z$  - locations in the same region.

It is to be noted that even a scalar surface impedance  $Z_s$  (for which  $\underline{Z} = \underline{1}_t Z_s$ ,  $Z_s =$  constant) couples the o-and e-modes (see Eqs. (4c, d)). Only when  $Z_s = 0, \infty$  (short or open circuit) or when  $\sigma = 0$  (normally incident plane wave) does the coupling disappear.

It is frequently more convenient to deal with a traveling wave representation involving incident and reflected waves rather than with the standing wave representation employed above. The required transformations, well-known in linear network theory,<sup>32</sup> are summarized below. Let

$$\bar{V} = \bar{a} + \bar{b}, \quad \bar{I} = \hat{Y}(\bar{a} - \bar{b}), \quad (16)$$

where the wave vectors  $\bar{a}$  and  $\bar{b}$  distinguish the amplitudes of waves traveling to the right and left, respectively (see Fig. 3),

$$\bar{a} \rightarrow \begin{bmatrix} a_o \\ a_e \end{bmatrix}, \quad \bar{b} \rightarrow \begin{bmatrix} b_o \\ b_e \end{bmatrix}, \quad (16a)$$

and  $\hat{Y}$  is the characteristic admittance matrix (see (12b)). Then the scattering transfer matrix  $T_s$  provides the direct connection between the incident and reflected wave amplitudes at terminals  $\alpha$  and  $\beta$  as follows

$$\begin{bmatrix} \bar{b}(\alpha) \\ \bar{a}(\alpha) \end{bmatrix} = T_s \begin{bmatrix} \bar{b}(\beta) \\ \bar{a}(\beta) \end{bmatrix}, \quad T_s \rightarrow \begin{bmatrix} \tau_{\alpha\alpha} & \tau_{\alpha\beta} \\ \tau_{\beta\alpha} & \tau_{\beta\beta} \end{bmatrix}, \quad (17)$$

where the scattering transfer matrix  $T_s$  is related to the impedance transfer matrix  $T_z$  in (11) via the linear transformation

$$T_s = \frac{1}{2} \begin{bmatrix} \hat{1} & -\hat{Z}(\alpha) \\ \hat{1} & \hat{Z}(\alpha) \end{bmatrix} T_z \begin{bmatrix} \hat{1} & \hat{1} \\ -\hat{Y}(\beta) & \hat{Y}(\beta) \end{bmatrix}. \quad (18)$$

$\hat{1}$  denotes the 2x2 unit matrix. When applied to an interface between two anisotropic media,  $t_{\alpha\beta} = t_{\beta\alpha} = 0$  from (11b), and (18) yields the following expressions for the 2x2 submatrices  $\tau_{ij}$ :

$$\tau_{\alpha\alpha} = \frac{1}{2} \left[ t_{\alpha\alpha} + \hat{Z}(\alpha) t_{\beta\beta} \hat{Y}(\beta) \right] = \tau_{\beta\beta}, \quad (19a)$$

$$\tau_{\alpha\beta} = \frac{1}{2} \left[ t_{\alpha\alpha} - \hat{Z}(\alpha) t_{\beta\beta} \hat{Y}(\beta) \right] = \tau_{\beta\alpha}. \quad (19b)$$

For a slab of length  $d$ , the scattering transfer matrix  $\hat{T}_s$  corresponding to  $\hat{T}_z$  in (12) has the simple diagonal representation

$$\hat{T}_s \rightarrow \begin{bmatrix} \tau_{11} & 0 \\ 0 & \tau_{22} \end{bmatrix}, \quad \tau_{11} = e^{-j\hat{\gamma}d}, \quad \tau_{22} = e^{+j\hat{\gamma}d}, \quad (20)$$

with  $\hat{\gamma}$  defined in (12b). Like the impedance transfer representation, the scattering transfer formulation is well suited to the analysis of cascaded networks as encountered in stratified media.

If the sources are located in a semi-infinite medium, the reflection phenomena are analyzed conveniently in terms of a scattering matrix representation which expresses the outgoing waves at terminals  $(\alpha, \beta)$  in terms of the incoming waves via the scattering matrix  $\mathcal{S}$ :

$$\begin{bmatrix} \bar{b}^{(\alpha)} \\ \bar{a}^{(\beta)} \end{bmatrix} = \mathcal{S} \begin{bmatrix} \bar{a}^{(\alpha)} \\ \bar{b}^{(\beta)} \end{bmatrix}, \quad \mathcal{S} \rightarrow \begin{bmatrix} s_{\alpha\alpha} & s_{\alpha\beta} \\ s_{\beta\alpha} & s_{\beta\beta} \end{bmatrix}. \quad (21)$$

The scattering matrix elements are related to those of the  $T_s$  matrix via

$$s_{\alpha\alpha} = \tau_{\alpha\beta} \tau_{\beta\beta}^{-1}, \quad s_{\alpha\beta} = \tau_{\alpha\alpha} - \tau_{\alpha\beta} \tau_{\beta\beta}^{-1} \tau_{\beta\alpha}, \quad s_{\beta\alpha} = \tau_{\beta\beta}^{-1}, \quad s_{\beta\beta} = -\tau_{\beta\beta}^{-1} \tau_{\beta\alpha}. \quad (22)$$

The above equations can also be used to describe an interface between an anisotropic and an isotropic medium. In this instance, it is more convenient to employ in the isotropic medium the linearly polarized E and H modes described in Appendix A.\*

(See Fig. 3 in Appendix C)

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\*Explicit expressions for the various coupling matrix elements, obtained after carrying out the operations and substitutions in the text, are given in reference 8. These rather cumbersome formulas are not listed here because of space limitations.

### III. Radiation condition and specification of $\mathcal{X}$

It was noted in connection with Eq. (8a) that  $\mathcal{X}_{o,e}$  is a multivalued function of  $(\xi^2 + \eta^2)^{1/2} = \sigma$ , and that its analytic properties must be specified if the integral representations (7a, b) for the fields are to be rendered unique.\* The definitions

$$\mathcal{X}_o \equiv \sqrt{U + \sqrt{U^2 - W}} - \sqrt{U + (\epsilon_1 - 1) \Delta/2}, \quad (23a)$$

$$\mathcal{X}_e \equiv \sqrt{U - \sqrt{U^2 - W}} = \sqrt{U - (\epsilon_1 - 1) \Delta/2}, \quad (23b)$$

with  $U$ ,  $W$ , and  $\Delta$  specified in (8b) and (4e), respectively, imply that  $\mathcal{X}_o$  has branch point singularities at  $U^2 = W$  ( $\Delta=0$ ) and at those values of  $\sigma$  for which  $\mathcal{X}_o = 0$ , while  $\mathcal{X}_e$  has branch point singularities at  $\Delta = 0$  and at those values of  $\sigma$  for which  $\mathcal{X}_e = 0$ . Although the branch point at  $\Delta = 0$  occurs in both the ordinary and extraordinary integrals in (7a, b)\*\* , the sum of the ordinary and extraordinary integrands is an even function of  $\Delta$  and therefore single-valued at  $\Delta = 0$ . This evenness, resulting from the fact that all ordinary quantities differ from the corresponding extraordinary ones only by the algebraic sign of  $\Delta$  (see (4c, d), (8a, b) and (9)), implies that a series expansion of the combined integrand about  $\Delta = 0$  contains only even powers of  $\Delta$ , whence  $\Delta = 0$  is a regular point. Thus, no special care need be taken in the definition of  $\mathcal{X}_{o,e}$  at  $\Delta = 0$  as regards the total integrand; if the ordinary and extraordinary integrals are treated separately, one may introduce convenient branch cut configurations relative to the branch points at  $\Delta = 0$  which render the integrands single-valued on a Riemann surface associated with this singularity.

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\* In plane stratified media,  $V_{o,e}$  and  $I_{o,e}$  may also possess pole singularities on the integration path. For the case of a single interface, these (surface wave) poles have been discussed in reference 8.

\*\*The two terms in the integrand of (7a) give rise to separate contributions which will be called ordinary (subscript o) and extraordinary (subscript e), respectively. Analogous considerations apply to (7b).

For a region comprising a series of slabs of finite width along the  $z$ -direction, the voltage and current solutions may be even functions of the propagation constants  $\mathcal{K}^{(a)}$  in each slab. In this case, the points where  $\mathcal{K}_o^{(a)} = 0$  and  $\mathcal{K}_e^{(a)} = 0$ , are regular. The situation is different, however, if one of the regions extends to  $z = \pm \infty$ . In this instance, the integrands are not even functions of the appropriate propagation constant, and the matter of the definition of  $\mathcal{K}_{o,e}$  must be studied in detail. The investigation is directly connected with the specification of the boundary condition at  $|z| \rightarrow \infty$ .

Consider a homogeneous anisotropic region which occupies the half space  $z_1 < z < \infty$ , and assume that all sources are located in the space  $z < z_2$  where  $z_2 > z_1$ . Then the solution of the transmission line equations (8) at points  $z > z_2$  comprises traveling waves characterized by the functions  $\exp(-jk\mathcal{K}_{o,e}z)$ , where  $k$  and  $\mathcal{K}_{o,e}$  have the values appropriate to the region in question, and  $k$  is assumed to be positive.\* The integration in (7a, b) extends over all real values of  $\sigma$  (see Appendix B). One notes from (23a, b) that  $\mathcal{K}_o$  and  $\mathcal{K}_e$  may be either real, imaginary, or complex, depending on whether  $\mathcal{K}_{o,e}^2$  is positive, negative, or complex (the latter case obtains when  $U^2 < W$ , i.e.,  $\Delta$  is imaginary). For those values of  $\sigma$  for which  $\mathcal{K}_o$  or  $\mathcal{K}_e$  has a non-vanishing imaginary part, the requirement

$$\text{Im } \mathcal{K}_{o,e} < 0 \quad (24)$$

assures that the fields remain finite as  $z \rightarrow \infty$  and serves to define the multi-valued functions  $\mathcal{K}_{o,e}$ . The associated mode fields decay exponentially with increasing  $z$  and represent a non-propagating wave.

If  $\mathcal{K}_o$  or  $\mathcal{K}_e$  is real, the  $\exp(-jk\mathcal{K}_{o,e}z)$  solution represents a propagating plane wave. Since all sources are confined to the region  $z < z_2$ , there must be a net flow of power toward  $z = \infty$  and it appears plausible to suppose that the power flow vector for each constituent propagating plane wave likewise has a component along the  $+z$  direction. It will now be shown that the latter condition follows from the former, i.e., that net total power flow toward  $z = \infty$  implies that each propagating plane wave carries power in this direction. This radiation condition (see also reference 7) then permits the unique determination of  $\mathcal{K}_{o,e}$  over the range of  $\sigma$  for which the propagation constants are real. It must be emphasized that the "energy radiation condition" is distinct from a "phase radiation condition" since the directions of phase and energy propagation in an anisotropic medium are generally different.

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\* In view of the remarks at the end of Appendix A, the conclusions derived herein can easily be adapted to imaginary  $k$ . In that case,  $\exp(-jk\mathcal{K}z) \rightarrow \exp(-j|k|\mathcal{K}z)$ , etc.

The requirement of net outward average power flow  $P_c$  through a plane at  $z = c > z_2$ , where  $c$  is a real constant, can be phrased as

$$P_c = \operatorname{Re} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \underline{z}_o \cdot \underline{S}(x, y, c) \geq 0, \quad \underline{S} = \underline{E}_t^* \times \underline{H}_t, \quad (25)$$

where  $\underline{S}$  is the complex Poynting vector. Upon substituting the integral representations (7a, b) into (25), interchanging the orders of integration and recalling the orthogonality relation (6), one obtains:

$$P_c = \operatorname{Re} \left\{ \int \int_{\mathcal{K}_{\text{real}}^2} d\xi d\eta \left[ V_o^* I_o + V_e^* I_e \right] + \int \int_{\mathcal{K}_{\text{complex}}^2} d\xi d\eta \left[ V_o^* I_e + V_e^* I_o \right] \right\} \geq 0, \quad (26)$$

where the first integral extends over that portion of the  $\xi - \eta$  plane for which  $\mathcal{K}_{o,e}^2$  is real ( $\Delta$  real), while the second extends over the remaining region wherein  $\mathcal{K}_{o,e}^2$  is complex ( $\Delta$  imaginary). Since

$$V_{o,e}(z) = C_{o,e} e^{-jk \mathcal{K}_{o,e}^2 z}, \quad z > z_2, \quad (27a)$$

where  $C_o$  and  $C_e$  are constants, one notes from the transmission line equations (8) (or from the equivalent network picture in Fig. 1(b)), that

$$I_{o,e}(z) = Y_{o,e} V_{o,e}(z), \quad z > z_2, \quad (27b)$$

so that (26) can be written as:

$$P_c = \operatorname{Re} \left\{ \int \int_{\mathcal{K}_{\text{real}}^2} d\xi d\eta \left[ Y_o |V_o|^2 + Y_e |V_e|^2 \right] + \int \int_{\mathcal{K}_{\text{complex}}^2} d\xi d\eta \left[ Y_e V_e V_o^* + Y_o V_o V_e^* \right] \right\} \geq 0. \quad (28)$$

From (23a, b),

$$\mathcal{K}_o^2 = \mathcal{K}_e^{*2}, \quad \text{when } \Delta \text{ is imaginary}, \quad (29a)$$

so that  $\mathcal{K}_o = \pm \mathcal{K}_e^*$ . Since  $\mathcal{K}_o$  and  $\mathcal{K}_e$  must both have negative imaginary parts (from (24)), it follows that

$$\mathcal{K}_o = -\mathcal{K}_e^*, \quad \operatorname{Im} \mathcal{K}_{o,e} < 0, \quad \text{when } \Delta \text{ is imaginary}, \quad (29b)$$



whence, from (8a),  $Y_o = -Y_e^*$ . Hence, the integral over the range of complex  $\chi^2$ , which arises from a coupling of the ordinary and extraordinary mode energies, is imaginary and does not contribute to  $P_c$ . If  $\chi_o^2$  and (or)  $\chi_e^2$  is negative real, i. e.,  $\chi_o$  and (or)  $\chi_e$  is imaginary, then  $Y_o$  and (or)  $Y_e$  is also imaginary (see (8a)) and the resulting integral does not contribute to  $P_c$ . Hence, (28) can be simplified to:

$$P_c = \iint_{\chi_o \text{ real}} Y_o |V_o|^2 d\xi d\eta + \iint_{\chi_e \text{ real}} Y_e |V_e|^2 d\xi d\eta > 0, \quad (30)$$

where the equality has been omitted since we are considering propagating waves which carry energy along the  $+z$ -direction. The symbol "Re" is superfluous because the integrals are now real. Since the radiation condition (30) must be valid for arbitrary source distributions in the region  $z < z_2$ , (i. e., for arbitrary  $V_o$  and  $V_e$ ), each of the integrands must satisfy the inequality whence  $P_c > 0$  if, and only if,

$$Y_{o,e} > 0 \quad \text{when } \chi_{o,e} \text{ is real.} \quad (31)$$

Hence, as stated above, each propagating plane wave must individually satisfy the radiation condition.

(8a) can now be employed to determine the algebraic sign of  $\chi_{o,e}$  in (23a, b) when  $\chi_{o,e}$  is real. Since the characteristic admittances must be positive, the sign of  $\chi_{o,e}$  is identical with that of  $(\Delta^2 \pm \sigma^2 \Delta)$ . This condition evidently depends on the values of the constitutive parameters  $\epsilon_1$ ,  $\epsilon_2$ , and is investigated below in detail for a plasma medium. Eqs. (24) and (31) specify  $\chi_{o,e}$  uniquely for all real values of  $\xi$  and  $\eta$  (or  $\sigma$ ), and can be used to determine the analytic continuation of the function  $\chi_{o,e}(\sigma)$  around branch point singularities located on the real  $\sigma$  axis. Some general remarks can be made concerning the determination of the range of propagating modes for which

$$\chi_o^2 = U \pm \sqrt{U^2 - W} \quad (32)$$

is positive real. One evidently must have  $U^2 - W > 0$ . If  $W > 0$  for some value of  $\sigma$ , then  $\sqrt{U^2 - W} < |U|$  and  $\chi_{o,e}^2 \geq 0$  if  $U \geq 0$ ; thus, both the ordinary and extraordinary modes propagate when  $U > 0$ , and neither propagates when  $U < 0$ . If  $W < 0$ , then  $\sqrt{U^2 - W} > |U|$ ; in this instance, the ordinary mode propagates for  $U \geq 0$  while the extraordinary

mode does not. The points  $\sigma_{3,4}$  at which  $U^2 - W = 0$ , or  $\Delta = 0$ , are obtained from (4e) as

$$\sigma_{3,4}^2 = 2\delta \left[ \delta \pm \sqrt{\delta^2 - 1} \right]. \quad (33)$$

If  $\delta > 1$ , these points lie on the real  $\sigma$  axis and give rise to  $\sigma$ -intervals for which  $\chi_{o,e}$  is complex.

While the specification of the function  $\chi_{o,e}(\sigma)$ , when real, can be carried out analytically from a study of (31), there also exists a simple graphical procedure utilizing the dispersion curves  $\chi(\sigma)$  vs.  $\sigma$  which locate the real solutions of the dispersion relation in the  $\chi$ - $\sigma$  plane (see (A9) and (A15)). For a plasma medium under the influence of a steady external magnetic field, these dispersion curves fall into various separate categories (distinguished by different ranges of the values of the applied signal, plasma, and cyclotron frequencies) and have been investigated in great detail in connection with plane wave propagation in the ionosphere.<sup>2,3,33,34</sup> In particular, it has been shown that a plane wave characterized by the variation

$$e^{-jk \left[ \xi x + \eta y + \chi(\sigma) z \right]} = e^{-jk \underline{p} \cdot \underline{r}}, \quad (34)$$

where  $\underline{p} = \underline{x}_0 \xi + \underline{y}_0 \eta + \underline{z}_0 \chi$  is the wave normal, carries real power in a direction perpendicular to the dispersion curve at the point  $(\xi, \eta, \chi)$ , and that the angle between the real part of the complex Poynting vector  $\underline{S}$  and the wave normal is less than  $90^\circ$ .<sup>10,35</sup> The direction of energy flow is commonly called the "ray" direction, and the above-mentioned relations are schematized in Fig. 4 for a typical case for which the dispersion curve has a closed and an open branch. The dispersion curves (or refractive index curves\*) can be shown to be rotationally symmetric about the d.c.

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\*Upon writing  $\exp[-jk \underline{p} \cdot \underline{r}] = \exp[-j \underline{p} \underline{k} \cdot \underline{r}]$ , where  $k$  is the wave-number defined in (3d) and the vector  $\underline{k}$  is parallel to  $\underline{p}$ , one may interpret  $\underline{p}$  as the refractive index (with respect to a medium having a dielectric constant  $\epsilon = \epsilon_0 \epsilon_z$ ) for the wave traveling in the direction  $\underline{k}$ . Since  $\underline{p}^2 = \chi^2 + \sigma^2$ , the distance from the origin to the  $\chi$  vs.  $\sigma$  surface yields the magnitude of  $\underline{p}$  as a function of the polar angle  $\theta$  measured from the direction of the applied d.c. magnetic field. Hence, this surface also constitutes the "refractive index surface" for the medium; the latter designation is used extensively in ionospheric propagation theory.

(See Fig. 4 in Appendix C)

magnetic field direction ( $z$  axis) whence a cross-section in the  $x-z$  plane suffices. In Fig. 4, the spatial coordinate axes have been superposed upon the  $\chi - \xi$  axes, so that the vector directions of the wave normal and ray are directly those in the  $x-z$  plane.

The radiation condition (31) can now be interpreted as corresponding to those values of  $\chi_{o,e}$  and  $\sigma$  for which  $\text{Re } \underline{S}$  has a component in the  $+z$  direction. Since  $\underline{p} \cdot (\text{Re } \underline{S}) > 0$ , the pertinent segments of the dispersion curves are those shown shaded in Fig. 4. One notes that it is possible to have  $\underline{p} \cdot \underline{z}_0 = \chi < 0$  while  $(\text{Re } \underline{S}) \cdot \underline{z}_0 > 0$ , corresponding to a "backward" wave in which the directions of energy and phase propagation along  $z$  are opposite. While this simple graphical construction allows the identification of those portions of the dispersion curve which contribute propagating waves carrying power in the positive  $z$ -direction, it gives no information as to which curve segments correspond to  $\chi_o$  or  $\chi_e$  as defined in (23a, b). The assignment of the proper ordinary and extraordinary branches defined herein requires a further study of the behavior of the functions  $U$  and  $W$  as discussed in connection with (32). These remarks are developed further in the next section where we analyze in detail the nature of the dispersion curves, and the associated disposition of  $\chi_{o,e}$ , for a plasma medium.

#### IV. The function $\mathcal{K}(\sigma)$ for a plasma medium

An ionized plasma medium under the influence of a steady external magnetic field  $H_0$  along the  $z$ -axis can be characterized by the following constitutive parameters:<sup>2,3</sup>

$$\epsilon_1 = 1 + \frac{\omega_c^2 \omega_p^2}{(\omega_c^2 - \omega^2)(\omega^2 - \omega_p^2)}, \quad \epsilon_2 = \frac{\omega \omega_c \omega_p^2}{(\omega_c^2 - \omega^2)(\omega^2 - \omega_p^2)}, \quad \epsilon_z = 1 - \frac{\omega_p^2}{\omega^2}, \quad (35)$$

where  $\omega$ ,  $\omega_p$  and  $\omega_c$  are the angular applied, plasma, and cyclotron frequencies, respectively. In this simple description of a plasma, only the electrons are considered mobile, and collisions with ions and neutral particles are neglected; also, the impressed a. c. field amplitudes are required to be small. In terms of the electron density  $N$ , the electronic charge  $e$ , and the electronic mass  $m$ , one may express  $\omega_p$  and  $\omega_c$  as:

$$\omega_p^2 = \frac{eN}{m\epsilon_0}, \quad \omega_c = \frac{e\mu_0 H_0}{m}. \quad (35a)$$

To assess the properties of  $\mathcal{K}$  as a function of  $\sigma$ , it is necessary to investigate the behavior of  $U$  and  $W$  (see (8b) for  $\epsilon_z > 0$ ) for various values of  $\omega$ ,  $\omega_c$ ,  $\omega_p$ . To facilitate this analysis, the parameters  $\epsilon_1$ ,  $2\epsilon_1/(\epsilon_1 + 1)$ ,  $\sigma_2^2$ ,  $\sigma_3^2$ ,  $\sigma_4^2$  have been plotted in Fig. 5 as functions of  $\omega$ . Fig. 5(a) pertains to the case  $\omega_p < \omega_c$ , while  $\omega_p > \omega_c$  in Fig. 5(b). If a scale for  $\sigma^2$  is superimposed along the positive imaginary axis, the plot may also be used to exhibit the ranges of  $\sigma^2$  which, at a given frequency  $\omega$ , correspond to propagating (real  $\mathcal{K}$ ) or non-propagating (complex  $\mathcal{K}$ ) waves (see (32) et seq.). These propagating wave domains have been shaded in Fig. 5; no propagation obtains outside the shaded regions. Since the defining expressions for  $\mathcal{K}_{o,e}$  change when  $\epsilon_z < 0$ , i. e., when  $\omega < \omega_p$  (see (A13)), a separate scale for  $\sigma^2$  along the negative vertical axis is used in this range. The frequency domain is subdivided naturally into the various intervals exhibited in Fig. 5.  $\omega_1$  and  $\omega_4$  denote, respectively, the larger and smaller of the frequencies for which  $\sigma_2^2 = 0$ ,  $\omega_2$  corresponds to  $\epsilon_1 = 0$ , while  $2\epsilon_1/(\epsilon_1 + 1) = \sigma_2^2$  at  $\omega_3$ .

The behavior of  $\mathcal{K}_{o,e}$  for waves satisfying the radiation condition at  $z \rightarrow +\infty$  in the various frequency ranges is summarized in Table I. The first column lists the branch point singularities which lie on the positive real  $\sigma$ -axis in the ordinary and extraordinary integrals. The notation  $a^+$  or  $a^-$  signifies that a branch point lies at  $\sigma = a$ , and that the path of integration is indented around it into the upper or lower half of the  $\sigma$ -plane, respectively. To each  $a^+$  there corresponds a  $(-a)^+$ , i. e., a branch point on the negative real axis, which is not listed explicitly.

Typical dispersion curves in the various frequency ranges are shown in the left half of Fig. 6. These curves represent the real  $\mathcal{K}$  -  $\sigma$  solutions of Eqs. (A9) or (A13) and the portions corresponding to the ordinary and extraordinary waves, as defined in (23a, b), are labeled "o" and "e", respectively. \* (If no curve is shown, the corresponding wave type does not propagate in this frequency range). The singularities of  $\mathcal{K}_o(\sigma)$  and  $\mathcal{K}_e(\sigma)$  along the real  $\sigma$ -axis are exhibited in the right half of the figure. (The branch points at  $\sigma_{3,4}$  in (33) lie on the real axis only when  $\delta = \epsilon_2/(\epsilon_1 - 1) = \omega/\omega_c > 1$ ). On the darkened segments,  $\mathcal{K}$  is real; elsewhere it is complex. The proper continuation of the functions  $\mathcal{K}_{o,e}(\sigma)$  around the various branch point singularities is effected in accord with conditions (24) and (31), as summarized in Table I. To render these functions unique in the multisheeted complex  $\sigma$ -plane, branch cuts have been drawn. This partitioning removes any ambiguities as to the disposition of the integration path with respect to singularities of  $\mathcal{K}_{o,e}$ ; the path is distorted around the branch points in a manner illustrated in Fig. 6(a) (a transformation of variables in (7a, b) from  $\xi - \eta$  to  $\sigma$  can be accomplished via (B1)). \*\* The integrands in (7a, b) will generally also contain pole singularities on the real  $\sigma$ -axis which must be investigated separately. Such an investigation has been carried out in connection with the problem of radiation from an anisotropic half space (see reference 8). The discussion herein suffices for the complete analysis of radiation in an infinite anisotropic medium where pole singularities do not arise. Application of these results to the detailed study of the far field radiated by a dipole source is to form the subject of a separate publication (see also reference 8).

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\* The asymptotes of the open branches of the dispersion curves (Figs. 6c, d, f, i) can be found by looking for real  $\mathcal{K}$ ,  $\sigma$  solutions in (A9) and (A15) as  $\mathcal{K}, \sigma \rightarrow \infty$ . Solutions exist only when  $\epsilon_1 < 0$ , in which instance  $\mathcal{K} \rightarrow \pm \sqrt{\epsilon_1} \sigma$  as  $\sigma \rightarrow \infty$ . The open branches arise for the ordinary and extraordinary modes when  $\epsilon_z > 0$  and  $\epsilon_z < 0$ , respectively.

\*\* If the range of propagating modes extends to  $\sigma \rightarrow \pm \infty$  (see Figs. 6c, d, f, i), absolute convergence of the integrals can be secured by deforming the end points of the integration path away from the real  $\sigma$ -axis into a region where the integrands decay exponentially.

Table I

Case		Singularities (on real axis)	Range of propaga- ting modes	Sign of $\chi$ (when real)
A	o	$1^+, \sigma_3^-, \sigma_4^+$	$\sigma^2 < 1$	$\chi_o > 0$
	e	$\sigma_2^+, \sigma_3^-, \sigma_4^+$	$\sigma^2 < \sigma_2^2$	$\chi_e > 0$
B	o	$1^+, \sigma_3^-, \sigma_4^+$	$\sigma^2 < 1$	$\chi_o > 0$
	e	$\sigma_3^-, \sigma_4^+$	none	
C	o	$1^+, \sigma_3^-, \sigma_4^+, \sigma_2^+$	$\sigma^2 < 1, \sigma_2^2 < \sigma^2 < \infty$	$\chi_o > 0, \chi_o < 0$
	e	$\sigma_3^-, \sigma_4^+$	none	
D	o	$1^+, \sigma_3^-, \sigma_4^+$	$\sigma^2 < 1, \sigma_4^2 < \sigma^2 < \infty$	$\chi_o > 0, \chi_o < 0$
	e	$\sigma_3^-, \sigma_4^+, \sigma_2^+$	$\sigma_4^2 < \sigma^2 < \sigma_2^2$	$\chi_e > 0$
E	o	$\sigma_2^+$	$\sigma^2 < \sigma_2^2$	$\chi_o > 0$
	e	$1^+$	$\sigma^2 < 1$	$\chi_e > 0$
F	o	$ \sigma_2 ^+$	$\sigma^2 <  \sigma_2^2 $	$\chi_o > 0$
	e	none	$\sigma^2 < \infty$	$\chi_e > 0$
G	o	none	none	
	e	$ \sigma_2 ^+$	$\sigma^2 <  \sigma_2^2 $	$\chi_e > 0$
H	o	$\sigma_3^-, \sigma_4^+$	none	
	e	$\sigma_3^-, \sigma_4^+$	none	
I	o	none	none	
	e	none	$\sigma^2 < \infty$	$\chi_e > 0$

Appendix A.Determination of the vector eigenfunctions

The transverse electromagnetic fields excited by arbitrary electric or magnetic current distributions in the anisotropic medium are to be represented as a superposition of transverse vector modal solutions which individually satisfy the source-free field equations and the boundary conditions in the transverse domain. To eliminate the  $z$ -dependence from (3a, b), characterized by the translation operator  $\partial/\partial z$ , we assume a separable representation of the form

$$\underline{E}_{ti}(\underline{r}) = \underline{\bar{e}}_i(x, y) e^{-j k \mathcal{K}_i z}, \quad k = k_0 \sqrt{\epsilon_z} > 0, \quad (\text{A1a})$$

$$\underline{H}_{ti}(\underline{r}) = Y_i \underline{\bar{h}}_i(x, y) e^{-j k \mathcal{K}_i z} \quad (\text{A1b})$$

where  $\mathcal{K}_i$  and the normalization parameter  $Y_i$  are constant, and the subscript  $i$  represents the modal index. Eqs. (3a, b) then define the vector eigenvalue problem for the transverse mode functions  $\underline{\bar{e}}_i$  and  $\underline{\bar{h}}_i$ :

$$\mathcal{K}_i \underline{\bar{e}}_i = Y_i \xi \left[ \underline{1}_t + \frac{\nabla_t \nabla_t}{k^2} \right] \cdot \left( \underline{\bar{h}}_i \times \underline{z}_0 \right), \quad \xi = \sqrt{\frac{\mu_0}{\epsilon}}, \quad (\text{A2a})$$

$$Y_i \mathcal{K}_i \xi \underline{\bar{h}}_i = \left[ \underline{1}_t + \frac{\nabla_t \nabla_t}{k^2} \right] \cdot \left( \underline{z}_0 \times \underline{\bar{e}}_i \right). \quad (\text{A2b})$$

Since the transverse ( $x - y$ ) domain is unbounded, the eigenfunctions have a plane wave dependence characterized by

$$\underline{\bar{e}}_i = \frac{k}{2\pi} \underline{e}(\xi, \eta) e^{-j k (\xi x + \eta y)}, \quad \underline{\bar{h}}_i = \frac{k}{2\pi} \underline{h}(\xi, \eta) e^{-j k (\xi x + \eta y)}, \quad (\text{A3})$$

where the continuous spectrum of the normalized wavenumbers  $\xi$  and  $\eta$  runs over the values  $-\infty < (\xi, \eta) < \infty$ , and the factor  $(k/2\pi)$  has been included

for convenience in normalization. Since the operator  $\nabla_t$  in (A2a, b) is now replaceable by  $-jk\underline{\sigma}$ ,  $\underline{\sigma} = (\underline{x}_0 \xi + \underline{y}_0 \eta)$ , one obtains the algebraic equations for the transverse position vectors  $\underline{e}$  and  $\underline{h}$  (for fixed  $\xi$  and  $\eta$ ),

$$\mathcal{K} \underline{e} = Y \xi \underline{R} \cdot (\underline{h} \times \underline{z}_0), \quad Y \xi \mathcal{K}(\underline{h} \times \underline{z}_0) = \underline{S} \cdot \underline{e}, \quad (\text{A4})$$

where  $\underline{R}$  and  $\underline{S}$  are self-adjoint 2 x 2 dyadics (note:  $\underline{\epsilon}_t = \underline{\epsilon}_t^+$ ),

$$\underline{R} = \underline{1}_t - \underline{\sigma} \underline{\sigma} = \underline{R}^+, \quad \underline{S} = \underline{\epsilon}_t - (\underline{z}_0 \times \underline{\sigma})(\underline{z}_0 \times \underline{\sigma}) = \underline{S}^+. \quad (\text{A4a})$$

The superscript<sup>+</sup> denotes the adjoint operator. Eqs. (A4) can be re-expressed as the eigenvalue problem

$$\underline{P} \cdot \underline{e} = \mathcal{K}^2 \underline{e}, \quad \underline{P}^+ \cdot (\underline{h} \times \underline{z}_0) = \mathcal{K}^2 (\underline{h} \times \underline{z}_0), \quad \underline{P} = \underline{R} \cdot \underline{S}. \quad (\text{A5})$$

It follows from the theory of linear operators that the two eigenvectors  $\underline{e}$  and  $\underline{h} \times \underline{z}_0$  belonging to the same eigenvalue  $\mathcal{K}^2$  satisfy a bi-orthogonality relation as in (5). The subscripts o and e are employed to distinguish the two possible eigenvalues, and associated eigenvectors, in (A5).

Eqs. (A5) are solved readily in a basis comprising the orthogonal position vectors  $\underline{\sigma}$  and  $\underline{\sigma} \times \underline{z}_0$ , wherein we represent

$$\underline{e} = a \underline{\sigma} + b \underline{\sigma} \times \underline{z}_0, \quad \underline{h} = c \underline{\sigma} + d \underline{\sigma} \times \underline{z}_0. \quad (\text{A6})$$

A straight-forward calculation leads to

$$\hat{P} \begin{bmatrix} a \\ b \end{bmatrix} = \mathcal{K}^2 \begin{bmatrix} a \\ b \end{bmatrix}, \quad \hat{P}^+ \begin{bmatrix} c \\ d \end{bmatrix} = \mathcal{K}^2 \begin{bmatrix} c \\ d \end{bmatrix}, \quad (\text{A7})$$

$$\hat{P} \rightarrow \begin{bmatrix} \epsilon_1 (1 - \sigma^2) & -j \epsilon_2 (1 - \sigma^2) \\ j \epsilon_2 & (\epsilon_1 - \sigma^2) \end{bmatrix}, \quad \sigma^2 = \xi^2 + \eta^2, \quad (\text{A7a})$$

from which one obtains the normalized eigenvectors in (4a, b).



The eigenvalues  $\mathcal{K}_{o,e}^2$  are the two solutions of the dispersion equation

$$\mathcal{K}(\mathcal{K}, \sigma) \equiv \det (\hat{\mathbf{P}} - \mathcal{K}^2) = 0 \quad (\text{A8})$$

or

$$\mathcal{K}^4 + \mathcal{K}^2 \left[ \sigma^2(\epsilon_1 + 1) - 2\epsilon_1 \right] + \epsilon_1 \sigma^4 - (\epsilon_1^2 - \epsilon_2^2 + \epsilon_1) \sigma^2 + \epsilon_1^2 - \epsilon_2^2 = 0, \quad (\text{A9})$$

which yields

$$\mathcal{K}_e^2 = U \pm \sqrt{U^2 - W} = U \pm (\epsilon_1 - 1)\Delta/2, \quad (\text{A10})$$

with  $U$ ,  $W$ , and  $\Delta$  defined in (8b) and (4e), respectively. Evidently,

$$\mathcal{K}_e^2 = \mathcal{K}_e^{*2} \quad \text{when } U^2 > W \text{ } (\Delta \text{ real}), \quad (\text{Alla})$$

and

$$\mathcal{K}_e^2 = \mathcal{K}_e^{*2} \quad \text{when } U^2 < W \text{ } (\Delta \text{ imaginary}). \quad (\text{Allb})$$

(A9) is the "Booker quartic" for the longitudinal wave number  $\mathcal{K}$  ( $\mathcal{K}$  is identical with Booker's  $q$ , save for the normalization to  $k$  instead of  $k_0$ ; i. e.,  $q = \sqrt{\epsilon} \mathcal{K}$ ) (see reference 3, Eq. (13.13), with  $\beta = \delta = 0$ ).

The subscripts  $o$  and  $e$  distinguish the two solutions corresponding to the  $+$  and  $-$  signs in (A10), respectively. The respective definitions "extraordinary" and "ordinary" are commonly applied in plasma theory to those real solutions of the dispersion equation which correspond to waves whose propagation in a direction transverse to the  $z$ -axis is, or is not, affected by the presence of the d. c. magnetic field (along  $z$ ).<sup>\*</sup> Transverse propagation corresponds to  $\mathcal{K} = 0$ ; for the ordinary mode one then has  $\sigma = 1$ , while for the extraordinary mode,  $\sigma \neq 1$ . It has been customary to label as "ordinary" and "extraordinary" those (real) branches of the  $\mathcal{K}$  vs.  $\sigma$  (dispersion) curves which do, or do not, pass through the points  $\mathcal{K} = 0$ ,  $\sigma = \pm 1$ . For our purposes, however, it is more significant to effect a definition on the basis of the analytic properties of the multivalued functions  $\mathcal{K}_o(\sigma)$  and  $\mathcal{K}_e(\sigma)$  as defined in (8a). The resulting apportionment, between  $\mathcal{K}_o$  and  $\mathcal{K}_e$ , of the real branches of the dispersion curves will then not necessarily coincide with that mentioned above (see Sec. IV). We have nevertheless retained the terminology "ordinary" and "extraordinary" for the  $o$ - and  $e$ -solutions, respectively, because the real solutions of  $\mathcal{K}_o$  do generally include the points  $\mathcal{K}_o = 0$ ,  $\sigma_o = \pm 1$ .

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\* When the magnetic field is absent,  $\epsilon_2 = 0$  and  $\epsilon_1 = 1$ .

From (A4) and the orthogonality relations (5), one deduces for the characteristic admittance  $Y$ :

$$Y_{o,e} = \frac{1}{Z_{o,e}} = \frac{\underline{e}_{o,e} \cdot \underline{S} \cdot \underline{e}_{o,e}}{\zeta \mathcal{K}_{o,e}} = \frac{\mathcal{K}_{o,e}}{\zeta \underline{h}_{o,e} \times \underline{z}_o \cdot \underline{R} \cdot \underline{h}_{o,e} \times \underline{z}_o} \quad (A12)$$

Substitution of (4d) into the last expression in (A12) yields formula (8a).

If one of the regions is isotropic,  $\epsilon_1 = 1$  and  $\epsilon_2 = 0$  so that  $\underline{\epsilon}_t = \underline{1}_t$ . In this instance,  $\mathcal{K}_o^2 = \mathcal{K}_e^2 = 1 - \sigma^2$ , and it is convenient to choose the conventional linearly polarized E mode (single primes) and H mode (double primes) eigenfunctions

$$\underline{e}' = \underline{h}' \times \underline{z}_o = \frac{\underline{\sigma}}{\sigma}, \quad \underline{e}'' = \underline{h}'' \times \underline{z}_o = \frac{\underline{\sigma} \times \underline{z}_o}{\sigma} \quad (A13)$$

It can then be shown that the resulting transverse field representation is still given by (7a, b) provided that we replace all subscripts o by single primes and all subscripts e by double primes.\* The E and H mode voltages  $V'$ ,  $V''$  and the currents  $I'$ ,  $I''$  satisfy the transmission line equations (8), with the characteristic impedances defined as

$$Z' = \frac{1}{Y'} = \frac{k\mathcal{K}}{\omega\epsilon}, \quad Z'' = \frac{1}{Y''} = \frac{\omega\mu}{k\mathcal{K}}, \quad \mathcal{K} = \sqrt{1 - \sigma^2} \quad (A14)$$

The equivalent network for the mode coupling produced by a plane interface between an isotropic and an anisotropic region can then be deduced by proceeding as in (10) - (11).

If  $\epsilon_z < 0$ , i. e.,  $k = -j|k|$ , the normalization of the transverse and longitudinal wave numbers  $\xi$ ,  $\eta$ , and  $\mathcal{K}$  is taken with respect to  $|k|$ . Hence,  $k$  in (A1a) and (A3) should be replaced by  $|k|$ , whence  $\zeta$  in (A2a) is defined as  $\sqrt{\mu_o/|\epsilon|}$ . The resulting eigenvalue problem is then found to be the same as in (A5) provided that one replaces  $\sigma$  by  $(+j\sigma)$  and  $\underline{S}$  by  $(-\underline{S})$ . The dispersion relation is now given by

$$\mathcal{K}^4 + \mathcal{K}^2 \left[ \sigma^2(\epsilon_1 + 1) + 2\epsilon_1 \right] + \epsilon_1 \sigma^4 + (\epsilon_1^2 - \epsilon_2^2 + \epsilon_1) \sigma^2 + \epsilon_1^2 - \epsilon_2^2 = 0, \quad (A15)$$

i. e., (A9) applies provided that  $\mathcal{K}^2$  and  $\sigma^2$  are replaced by  $(-\mathcal{K}^2)$  and  $(-\sigma^2)$ , respectively. Finally, the eigenvectors are given as in Eqs. (4), if  $k$  and  $\sigma$

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\*The validity of the formal expressions obtained via this replacement does not necessarily imply that the E and H modes correspond to the limiting values of the o- and e- modes, respectively, as  $\underline{\epsilon}_t \rightarrow \underline{1}_t$ .

are replaced by  $|k|$  and  $(+j\sigma)$ , respectively (note:  $\underline{\sigma} \rightarrow +j\underline{\sigma}$ ). In summary, formulas (7) - (9) appropriate to  $\epsilon_z < 0$  can be obtained from those for  $\epsilon_z > 0$  by letting  $\sqrt{\epsilon_z} \rightarrow -j|\sqrt{\epsilon_z}|$ ,  $\sigma \rightarrow j\sigma$ ,  $\chi \rightarrow j\chi$ .

Appendix B.Cylindrical wave representation of the fields

Instead of the plane wave representation in (7a, b), it is frequently more convenient to employ a cylindrical wave representation based on the well-known transformation

$$I = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta f(\sigma) e^{-jk(\xi x + \eta y) + jk(\xi x' + \eta y')} \\ = 2\pi \sum_{n=-\infty}^{\infty} e^{-jn(\phi - \phi')} \int_0^{\infty} \sigma f(\sigma) J_n(k\sigma\rho) J_n(k\sigma\rho') d\sigma, \quad k > 0, \quad (B1)$$

where

$$\sigma = \sqrt{\xi^2 + \eta^2}, \quad x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad x' = \rho' \cos \phi', \quad y' = \rho' \sin \phi'. \quad (B2)$$

Each of the constituent integrals in (7a, b) resulting upon substitution of (4c, d) can be reduced to the form I upon observing that the vector  $\underline{\sigma}$  can be replaced by the vector operator  $(j/k) \nabla_t$ , and that the differentiation and integration operations can then be commuted. The scalar constituents of  $\underline{e}_o, e$  and  $\underline{h}_o, e$  remaining in the integrands are functions of  $\sigma$  only. The voltage and current  $V_{o, e}$  and  $I_{o, e}$  are solutions of the differential equations (8). Since the parameters  $\mathcal{X}, Y, Z, i, v$  in this equation depend on  $(\xi, \eta)$  only through  $\sigma$ , as do the coupling matrices (11) and (15) descriptive of interface or terminal effects,  $V_{o, e}$  and  $I_{o, e}$  can be expressed as functions of  $\sigma$  only, and a cylindrical wave representation is effected as in (B1). The explicit appearance of the factor  $\exp[jk(\xi x' + \eta y')]$  in the integrand of (B1) results from consideration of a point current element located at  $\underline{r}' = (x', y', z') = (\rho', \phi', z')$  (cf. (9a, b)). The expression for a distributed source is obtained by integration over the source point coordinate  $\underline{r}'$ .

If  $f(\sigma) = f(-\sigma)$ , i. e.,  $f$  is an even function of  $\sigma$ , the  $\sigma$  integral can be converted to run over the entire real  $\sigma$ -axis:

$$2 \int_0^{\infty} \sigma f(\sigma) J_n(k\sigma\rho) J_n(k\sigma\rho') d\sigma = \int_{-\infty}^{\infty} \sigma f(\sigma) J_n(\sigma\rho_{<}) H_n^{(2)}(\sigma\rho_{>}) d\sigma. \quad (B3)$$

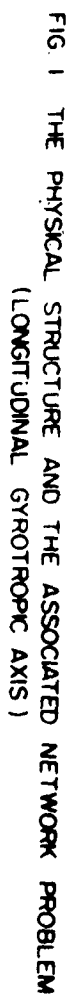
$\rho_{>}$  and  $\rho_{<}$  denote the greater and lesser of the variables  $\rho$  and  $\rho'$ , respectively. The second representation in (B3), involving an infinite integration contour, is particularly convenient for an asymptotic evaluation of the integral.

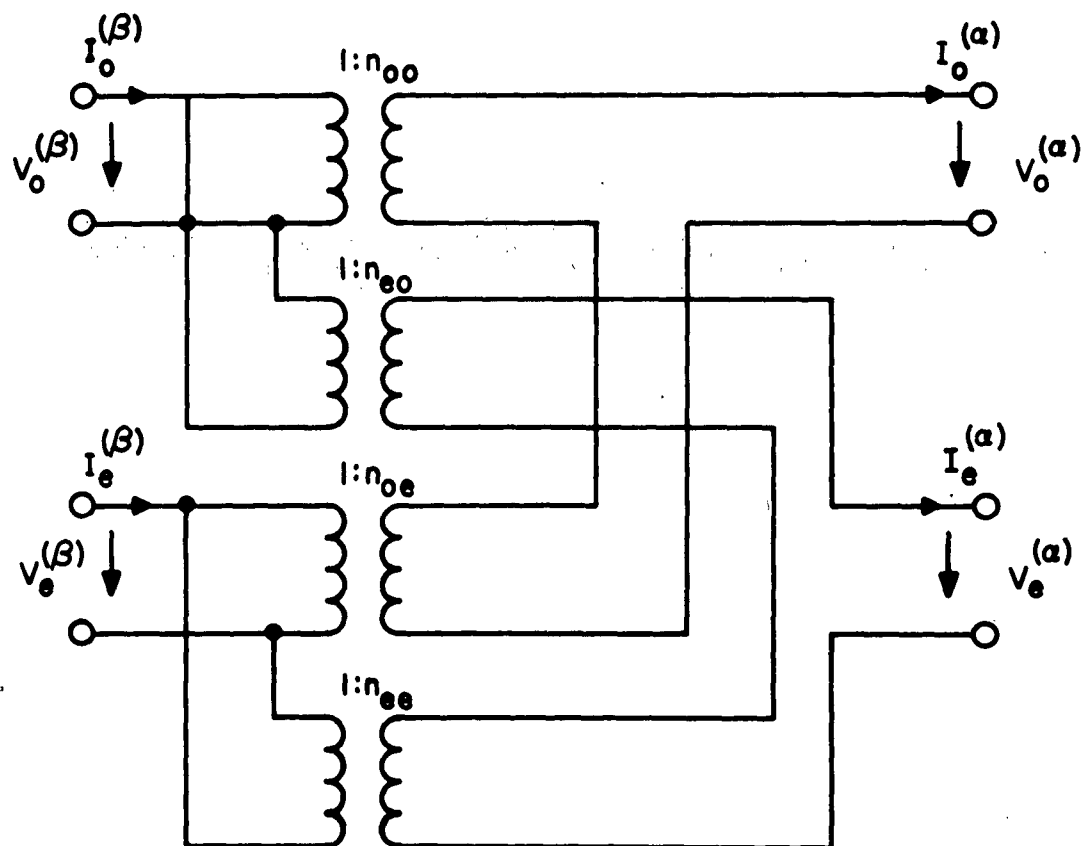
Appendix C

Figure Captions

- Fig. 1 - Physical structure and associated network problem
- Fig. 2 - Equivalent network for interface between two gyrotropic media
- Fig. 3 - Definition of traveling and standing wave quantities
- Fig. 4 - Dispersion curve, wave normal, and ray
- Fig. 5 - Frequency dependence of various plasma parameters
- Fig. 6 - Dispersion curves and singularities on  $\text{Re } \sigma$ -axis

The dispersion curves on the left show the behavior of  $\mathcal{X}$  vs.  $\sigma$  for propagating waves in various frequency ranges; the designations "o" and "e" distinguish the ordinary and extraordinary branches, respectively. Singularities due to  $\mathcal{X}_{o,e}(\sigma)$  on the real  $\sigma$ -axis are shown on the right. The dark segments denote propagating wave regions.





$$n_{lm} = \frac{k^{(\alpha)}}{k^{(\beta)}} \frac{h^{(\alpha)*}}{h^{(\beta)}} \times \frac{z_o}{z_m} \cdot \frac{e^{(\beta)}}{e^{(\alpha)}}, \quad l \rightarrow o, e, \quad m \rightarrow o, e$$

Fig. 2

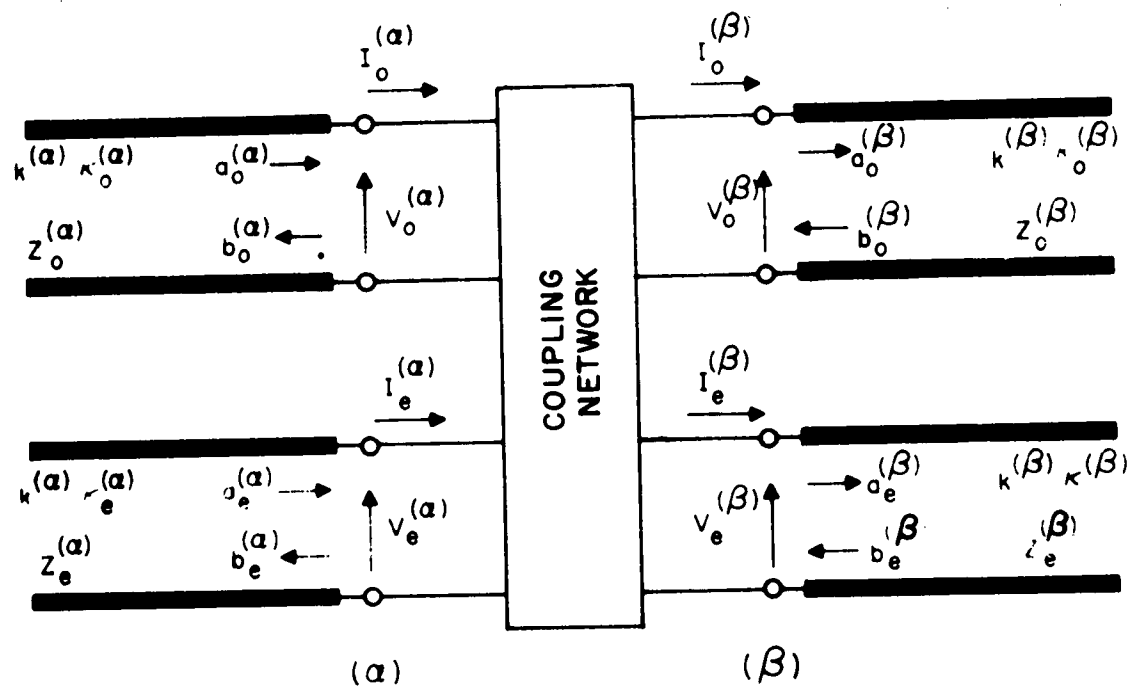
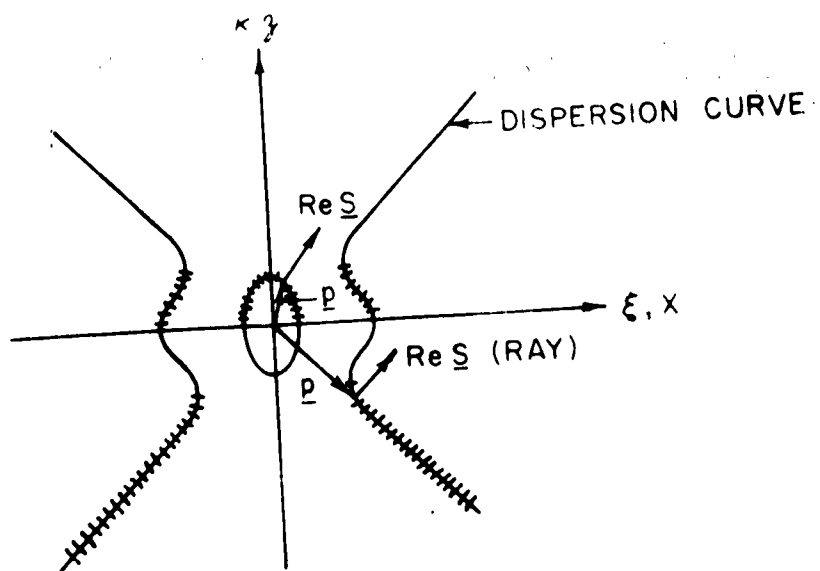


Fig. 3





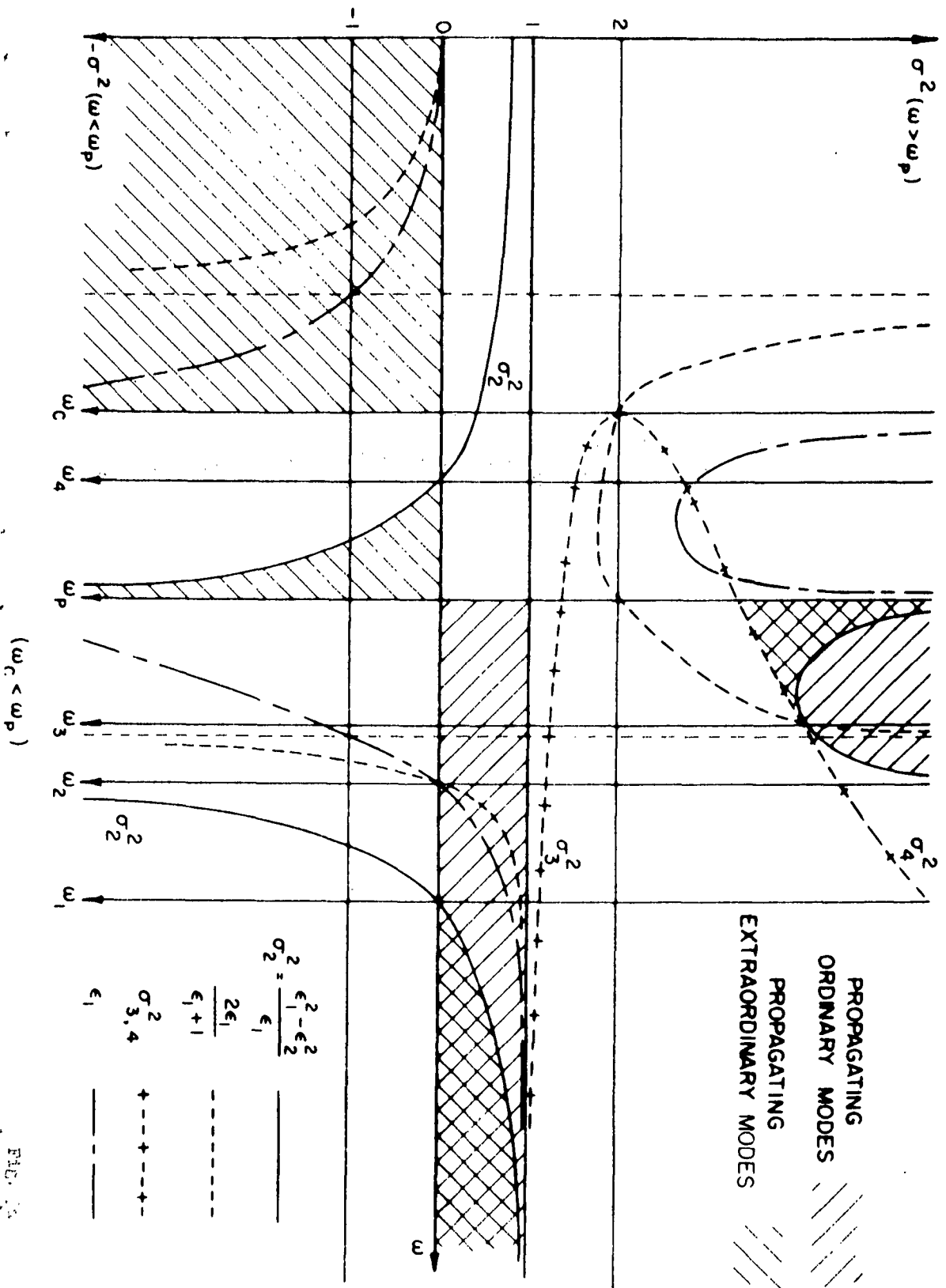


FIG. 2

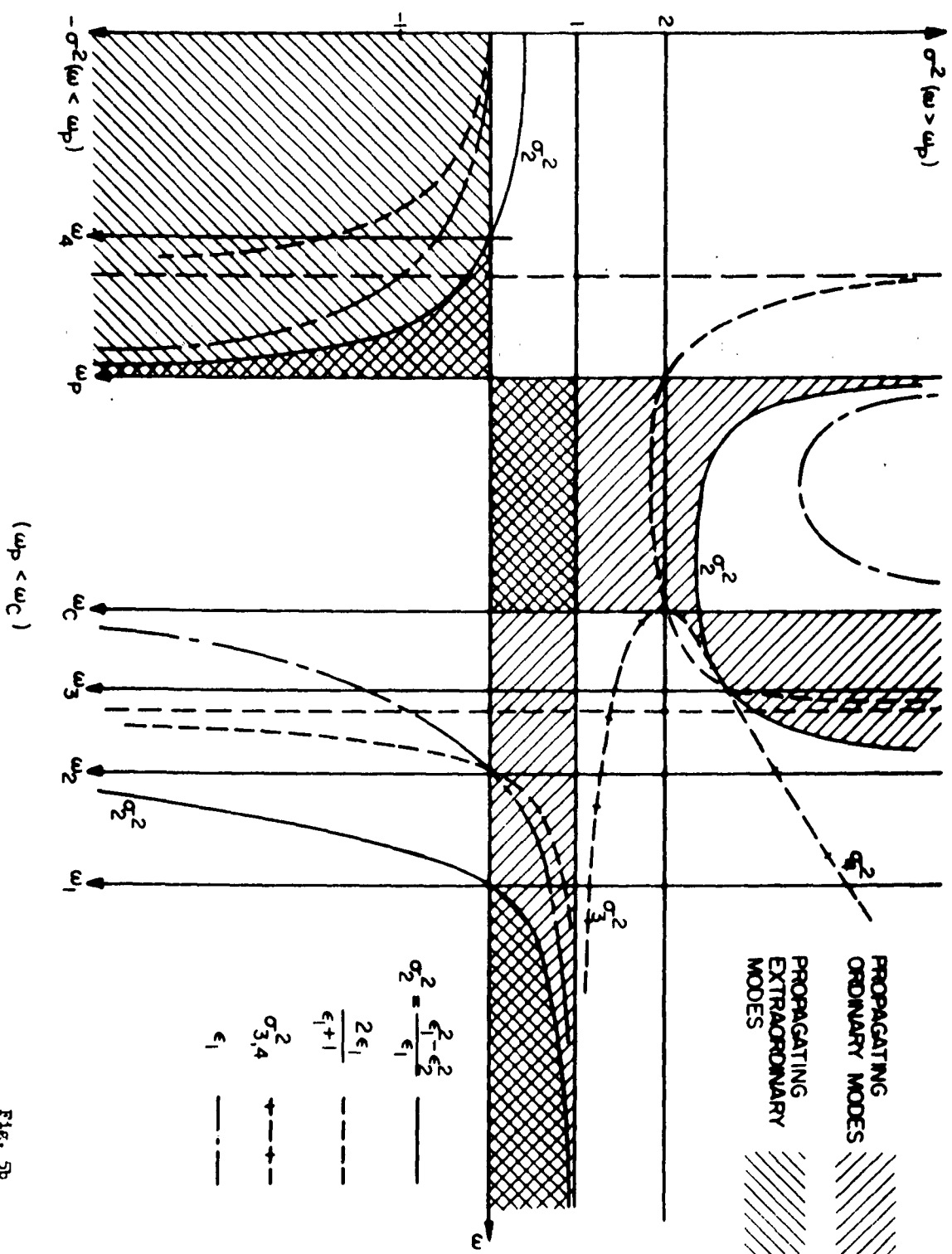
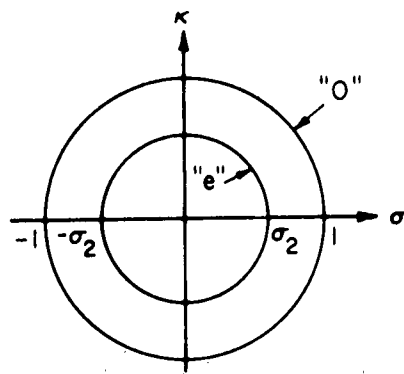
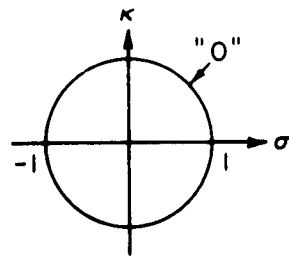
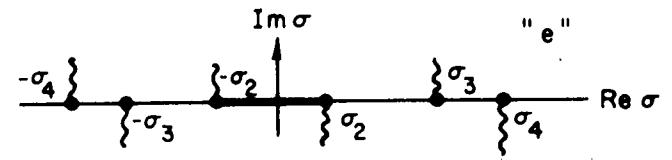
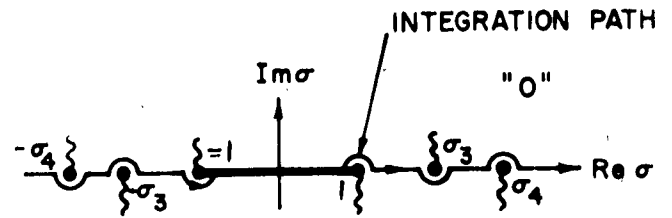


Fig. 5b

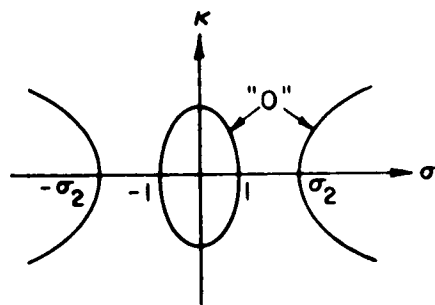
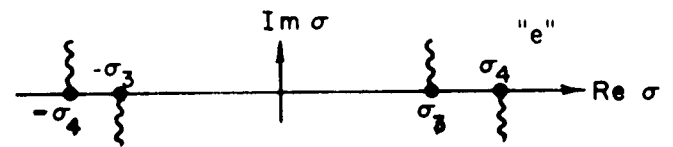
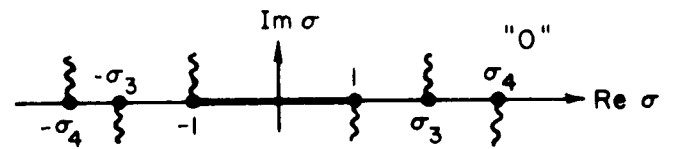
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$$a) \omega > \omega_1, \quad \omega_1 = \sqrt{\frac{\omega_c^2}{4} + \omega_p^2} + \frac{\omega_c}{2}$$



$$b) \omega_1 > \omega > \omega_2, \quad \omega_2 = \sqrt{\omega_p^2 + \omega_c^2}$$



$$c) \omega_2 > \omega > \omega_3, \quad \omega_3 = \sqrt{\frac{\omega_c^2}{2} + \sqrt{\frac{\omega_c^4}{4} + \omega_p^4}}$$

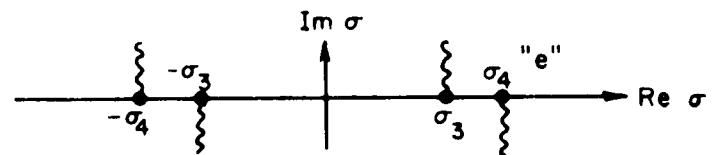
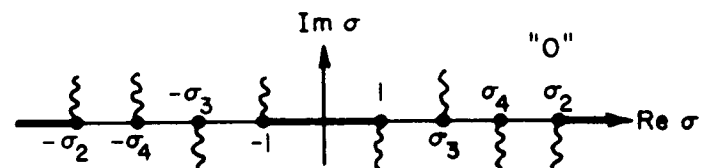
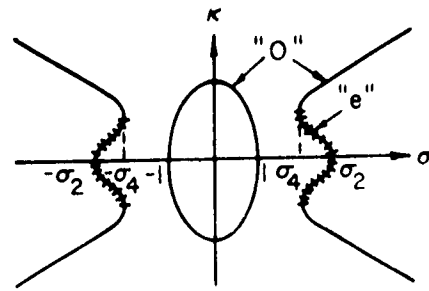
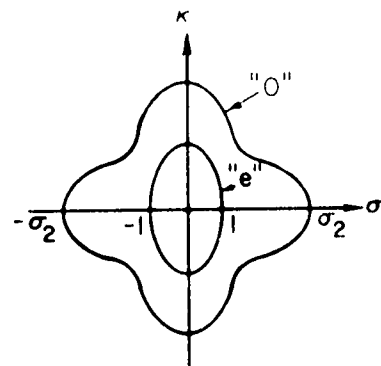
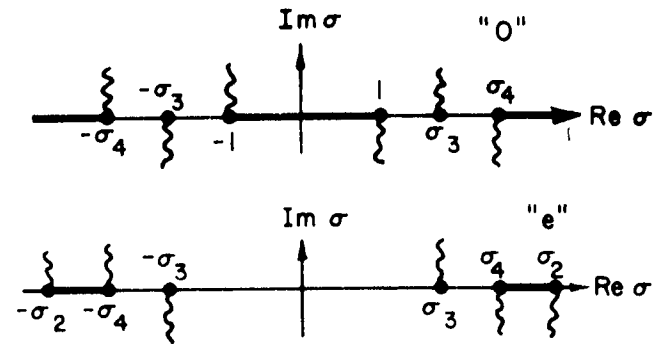


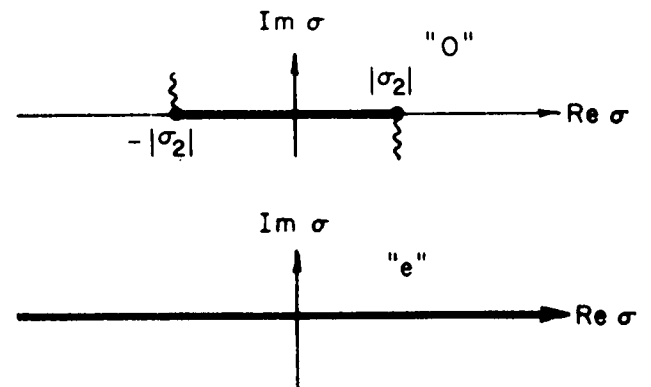
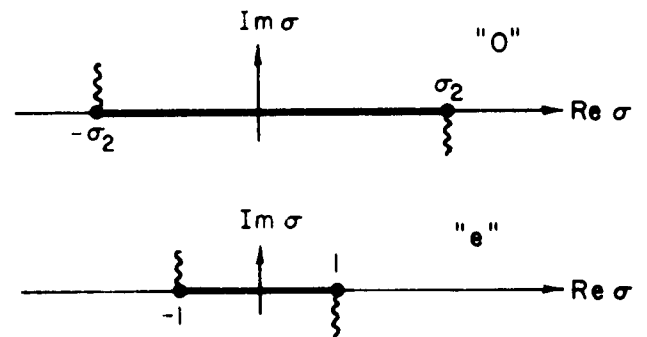
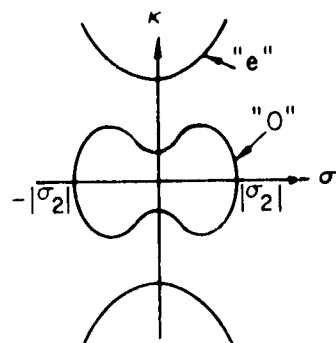
FIG. 6



d)  $\omega_3 > \omega > \max(\omega_p, \omega_c)$

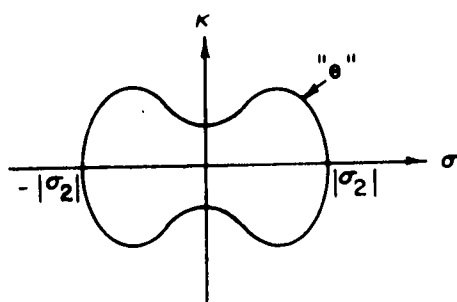


e)  $\omega_c > \omega > \omega_p$

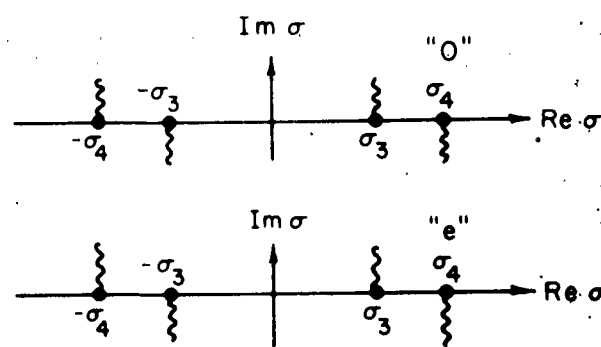
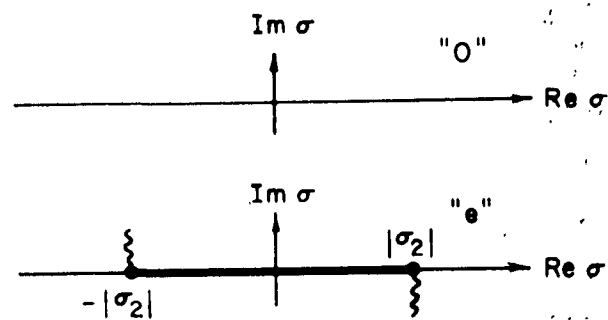
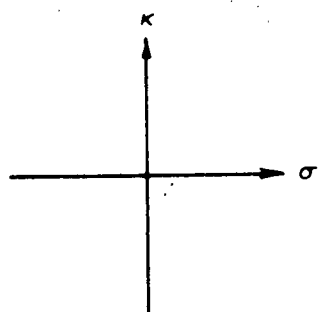


f)  $\min(\omega_p, \omega_c) > \omega > \omega_4$ ,  $\omega_4 = \sqrt{\omega_p^2 + \frac{\omega_c^2}{4}} - \frac{\omega_c}{2}$

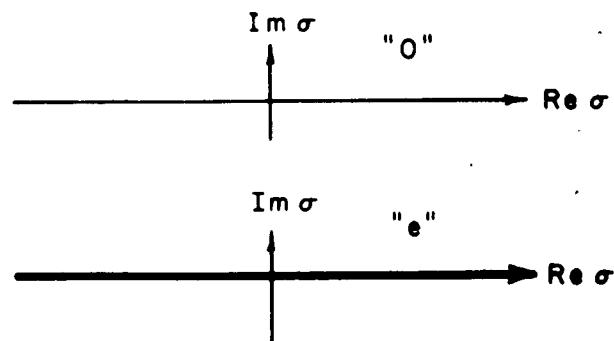
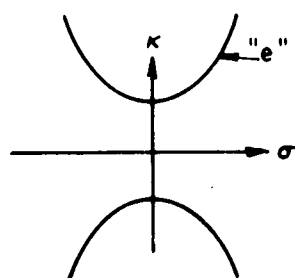
Fig. 6



g)  $\omega_p > \omega > \max(\omega_4, \omega_c)$



h)  $\omega_4 > \omega > \omega_c$



i)  $\min(\omega_4, \omega_c) > \omega > 0$

Fig. 6

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